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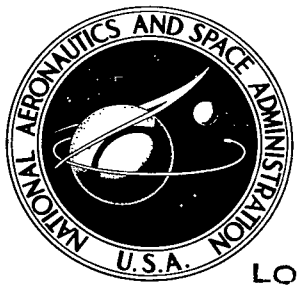

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A METHOD FOR THE ANALYSIS OF NONLINEARITIES IN AIRCRAFT DYNAMIC RESPONSE TO ATMOSPHERIC TURBULENCE

Kenneth Sidwell

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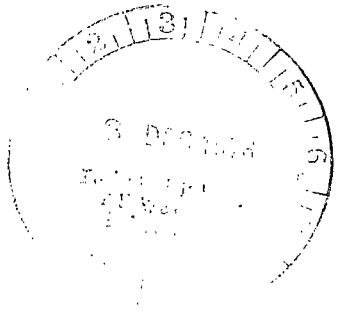

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16. Abstract

An analytical method is developed which combines the equivalent linearization technique for the analysis of the response of nonlinear dynamic systems with the amplitude modulated random process (Press model) for atmospheric turbulence. The method is initially applied to a bilinear spring system. The analysis of the response shows good agreement with exact results obtained by the Fokker-Planck equation. The method is then applied to an example of control-surface displacement limiting in an aircraft with a pitch-hold autopilot.

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A METHOD FOR THE ANALYSIS OF NONLINEARITIES IN AIRCRAFT DYNAMIC RESPONSE TO ATMOSPHERIC TURBULENCE

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SUMMARY

An analytical method is developed which combines the equivalent linearization technique for the analysis of the response of nonlinear dynamic systems with the amplitude modulated random process (Press model) for atmospheric turbulence. The method is initially applied to a bilinear spring system. The analysis of the response shows good agreement with exact results obtained by the Fokker-Planck equation. The method is then applied to an example of control-surface displacement limiting in an aircraft with a pitch-hold autopilot.

INTRODUCTION

Nonlinear effects can be important in aircraft dynamic response to atmospheric turbulence, particularly at design load levels. The importance of nonlinear effects is widely recognized, especially those influencing the action of stability augmentation and active control systems of aircraft in atmospheric turbulence. Structural design criteria based upon random process theory (the power spectral density method) require that the nonlinear effects of control system action upon loads at the limit load level be realistically or conservatively accounted for. (See refs. 1 and 2.) Current analysis procedures generally use analog

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simulation to study effects of nonlinearities upon the control system action. Several examples of the analysis of stability augmentation systems using analog simulation are the analyses for the B-52 (ref. 3), the B-70 (ref. 4), and the Lockheed 1011 (refs. 5 and 6). The study of reference 3, for example, shows that the displacement limiting of the control surface in severe turbulence has the effect of making the control system inoperative. This general effect is also shown by experimental data discussed in reference 7. Thus, nonlinear effects can be especially important in the use of control systems to alleviate turbulence-induced loads in aircraft.

A method for the analysis of nonlinear effects in aircraft dynamic response to atmospheric turbulence is developed in the present study. The method combines two basic analytical concepts: (1) the analysis of the response of nonlinear dynamic systems to random processes, and (2) the development of the amplitude modulated random process for modeling atmospheric turbulence. Exact solutions for the response of some nonlinear dynamic systems have been obtained by use of the Fokker-Planck equation (refs. 8 and 9). Approximate solutions have been obtained by the technique of equivalent linearization. The development and reviews of this method and its applications are given in references 10 to 12. The amplitude modulated random process, which is used to model atmospheric turbulence, was developed by Press and his associates (refs. 13 and 14). The mathematical properties of the amplitude modulated random process have been examined in references 15 and 16. The original contribution of the present study is the combination of these two separate subjects and the application of the resulting analytical method to one example of control limiting effects in aircraft dynamic response to atmospheric turbulence.

The present study is divided into two parts. The first part is the development of the response of a spring-mass-damper system with a bilinear stiffness force. The response of this nonlinear system to a Gaussian conditional process is developed by use of the Fokker-Planck equation. The response to the amplitude modu-

lated process is then developed by combining the response to the conditional process with the random variation of the amplitude parameter of the conditional process. The exact solution and two approximate solutions are developed. Since the bilinear spring system allows an exact solution, the adequacy of the approximate solutions for this type of nonlinear dynamic system can be evaluated. The second part of the study is an application of the equivalent linearization technique to one problem of nonlinear aircraft response to atmospheric turbulence. An example of control surface deflection limiting in an aircraft with an autopilot operating in a pitch-hold mode is considered.

SYMBOLS

A	standard deviation factor of conditional process
b	standard deviation of amplitude (σ) process
b_i	turbulence intensity parameter
C_0	normalization constant of probability density function
C_{m_δ}	aerodynamic moment coefficient for elevator deflection
$E[\ ]$	ensemble average
$E[\ \|\sigma\ ]$	ensemble average of conditional process (conditional on value σ)
erf( )	error function (ref. 17)
erfc( )	complementary error function, 1 - erf( )
$f(x)$	stiffness force

$g(\gamma,\eta)$ $= \gamma \exp[(\gamma^2 - 1)\eta^2]\operatorname{erfc}(\gamma\eta)$

$h(\ )$ nonlinear limiting function

h_e control effectiveness factor associated with nonlinear elevator deflection

K feedback gain for subscripted quantity

$K_0(\ )$ modified Bessel function of order zero (ref. 17)

k linear stiffness force coefficient

M_{δ} aerodynamic moment due to elevator deflection

m mass

$N(\ )$ exceedance expression or expected frequency of positive slope crossings of indicated level

N_0 expected frequency of positive slope crossings of zero level

P_i turbulence probability parameter

$p(\ )$ probability density function

$p(x|\sigma)$ conditional probability density function of X process conditional on value σ

q_d dynamic pressure

$S\bar{c}$ reference area times reference chord length

s	Laplace transformation variable; also amplitude parameter in appendix A
t	time
t_{ch}	servo system characteristic time
v	time derivative of x
x	displacement of bilinear spring system
x_b	breakpoint of bilinear stiffness force relation
$x_{m,\infty}$	limiting mean value of displacement of nonsymmetric bilinear spring system
x_0	$= (1 - \gamma^2)x_b$
α	parameter of nonsymmetric bilinear stiffness force relation: modulus of ratio of negative and positive displacement breakpoints
α_g	ratio of vertical component of turbulence velocity to aircraft forward speed
β	linear damping force coefficient
γ	$= \sqrt{\frac{k_1}{k_2}}$
$\delta(t)$	Dirac delta function
δ_c	commanded elevator deflection perturbation
δ_p	physical elevator deflection perturbation

δ_{pl} physical limit of elevator deflection

η $= x_b/\sqrt{2}A_{x1}\sigma$

θ pitch angle perturbation

$\xi(t)$ Gaussian white noise process

σ amplitude random process; amplitude parameter of conditional process; standard deviation of subscripted process, for example, σ_y means standard deviation of y

Subscripts:

am amplitude modulated process

b breakpoint

c conditional

dr Dempster-Roger method

e equivalent linearization

i,j,k,n integer index

p physical

1,2 regions of stiffness force relations

A tilde over a symbol denotes a variable of integration. Dots over a symbol denote derivatives with respect to time.

RESPONSE OF SYMMETRIC BILINEAR SPRING SYSTEM

The response of a specific nonlinear dynamic system to random excitation by the amplitude modulated process is analyzed in this section. The nonlinear system is one for which the exact solution of the system response to a stationary Gaussian process is known. The present development combines the analysis of the response of nonlinear dynamic systems by the Fokker-Planck equation with the development of the amplitude modulated random process. The amplitude modulated process, which is the combination of a local random process with a random amplitude modulation, is discussed in appendix A. The application of the Fokker-Planck equation to nonlinear dynamic systems is also discussed in appendix A. The solution for the response of the nonlinear dynamic system to the amplitude modulated process is developed in two steps. First, the solution is developed for excitation by the conditional process, which is a Gaussian process with a given value of amplitude parameter. The effects upon the response of varying the amplitude parameter of the excitation process are considered in detail. Second, the solution is developed for excitation by the amplitude modulated process, by using the results for the conditional process and introducing the random amplitude modulation through the random variation of the amplitude parameter of the conditional process.

Bilinear Spring System

A single-degree-of-freedom spring-mass-damper system with a nonlinear stiffness force is considered

$$m\ddot{x} + \beta \dot{x} + f(x) = \sigma \xi(t) \tag{1}$$

A specific nonlinear stiffness force, the symmetric combination of two regions of linear stiffness force, is considered. The resulting bilinear stiffness force relation is

$$f(x) = \begin{cases} k_2(x + x_0) & (x < -x_b) \\ k_1 x & (-x_b \leq x \leq x_b) \\ k_2(x - x_0) & (x_b < x) \end{cases} \qquad (2)$$

where

$$\gamma^2 = \frac{k_1}{k_2}$$

$$x_0 = (1 - \gamma^2)x_b$$

The stiffness force relation is plotted in figure 1 as a function of the displacement. The symmetric stiffness force has two (the inner and the outer) linear regions which intersect at the displacement breakpoint x_b.

The response of the bilinear spring system to random excitation is dominated by one of the two linear regions in the limiting response cases. In the limit of small displacements relative to the breakpoint, the response is dominated by the inner linear region of the bilinear stiffness force. The limit of small displacements corresponds to a large value of the breakpoint x_b or to a small value of the amplitude parameter σ or to both. In the limit of large displacements, the response is dominated by the outer linear region. Thus, the solution for the response of the nonlinear system to the random excitation has these two limiting cases plus intermediate cases where both regions of the stiffness force significantly influence the system response, as is discussed subsequently.

Conditional Process

The excitation process of the bilinear spring system (eq. (1)) is a Gaussian white noise process which is multiplied by an

amplitude parameter σ. The excitation process is accordingly considered to be a conditional process, that is, a Gaussian white noise process with a given value of the amplitude parameter. Thus, the excitation process has an arbitrary reference value of the power spectral density function which is multiplied by the square of the amplitude parameter. The dynamic response of the bilinear spring system is described in probabilistic terms by the joint probability density function of the system displacement and velocity, which is conditional on the value of the amplitude parameter. From this function the associated moments and exceedance expression for the response to the conditional process can be determined.

Exact solution.- The development of the joint probability density function of the system displacement and velocity from the application and solution of the Fokker-Planck equation is discussed in detail in appendix B. The displacement and velocity of the system response are independent for the conditional process

$$p(x,v|\sigma) = p(x|\sigma)\, p(v|\sigma) \tag{3}$$

where $v = \dot{x}$. Thus, the displacement and velocity can be considered separately. By use of the Fokker-Planck equation, the probability density function of the displacement is

$$p(x|\sigma) = \begin{cases} C_0 \exp\left\{-\dfrac{1}{2A_{x2}{}^2\sigma^2}\left[(x + x_0)^2 + \gamma^2 x_b x_0\right]\right\} & (x < -x_b) \\ C_0 \exp\left\{-\dfrac{1}{2A_{x1}{}^2\sigma^2}\, x^2\right\} & (-x_b \leq x \leq x_b) \\ C_0 \exp\left\{-\dfrac{1}{2A_{x2}{}^2\sigma^2}\left[(x - x_0)^2 + \gamma^2 x_b x_0\right]\right\} & (x_b < x) \end{cases} \tag{4}$$

where

$$\frac{A_{x2}^2}{A_{x1}^2} = \frac{k_1}{k_2} = \gamma^2 \tag{5}$$

The quantity C_0 is a normalization constant. The displacement is non-Gaussian because of the nonlinearity of the stiffness force. The probability density function consists of Gaussian regions which correspond to the linear regions of the symmetric stiffness force relation. The mean and variance of the distributions within these regions are those of the corresponding linear systems. The probability density function is Gaussian in the limits of zero and infinite values of the stiffness breakpoint x_b and has the functional forms for the corresponding limiting linear systems. The effects of the stiffness nonlinearity are shown in a concise form by the conditional variance of the displacement, which is calculated from the conditional probability function of the displacement (eq. (4)),

$$E[x^2|\sigma] = \sigma_{xc}^2 = A_x^2(\sigma)\sigma^2 \tag{6}$$

$$\frac{A_x^2(\sigma)}{A_{x1}^2} = 1 + \frac{(\gamma^2 - 1)\left\{-\frac{2}{\sqrt{\pi}}\eta(\gamma^2 - 1)e^{-\eta^2} + \left[1 + 2(\gamma^2 - 1)\eta^2\right]g(\gamma,\eta)\right\}}{\operatorname{erf}\eta + g(\gamma,\eta)} \tag{7}$$

where

$$\eta = \frac{x_b}{\sqrt{2}A_{x1}\sigma}$$

$$g(\gamma,\eta) = \gamma \exp\left[(\gamma^2 - 1)\eta^2\right] \operatorname{erfc}(\gamma\eta)$$

The conditional standard deviation is expressed in terms of the standard deviation factor A. This notation is convenient since the standard deviation factor is independent of the amplitude parameter σ for linear systems.

The dependence of the standard deviation factor of the displacement upon both the amplitude parameter σ and the stiffness displacement breakpoint x_b is shown in figures 2 and 3. Figure 2 shows the standard deviation factor of the displacement A_x (eq. (7)) in nondimensional form as a function of the amplitude parameter. The relationship is shown for the value of the parameter γ (eq. (5)) equal to two. In this case the incremental stiffness force is decreased and the associated standard deviation factor is increased in the outer linear region. The standard deviation factor of the displacement is dominated by the inner or outer regions of the stiffness force relation (eq. (2)) in the limits of very small or very large displacements, respectively. The standard deviation factor of the displacement approaches that of the inner linear system A_{x1} in the limit of small values of the amplitude parameter σ. Similarly, the factor approaches that of the outer linear system A_{x2} as the amplitude parameter becomes large. The same response pattern occurs in the limits of large and small values of the stiffness breakpoint x_b, respectively. Thus, the standard deviation factor of the displacement shows the two limiting cases of the bilinear spring system. The standard deviation factor of the displacement is plotted in a different, normalized form in figure 3 for several values of the parameter γ. Again, the standard deviation factor shows the two limiting cases and the intermediate states of the bilinear stiffness force relation. In normalized form the standard deviation factor is weakly dependent on the value of the parameter γ.

The standard deviation of the displacement is plotted in nondimensional form in figure 4 as a function of the amplitude parameter for one value of the parameter γ. The values shown in figure 4 correspond to those in figure 2; the two quantities are related by equation (6). For linear systems the relationship of

figure 4 is linear since the standard deviation factor is independent of the amplitude parameter. Deviations from a linear relation are due to the nonlinearity of the dynamic system. Also shown in figure 4 is the Dempster-Roger approximation for the standard deviation, which is discussed subsequently.

The probability density function of the velocity of the bilinear spring system is determined by the Fokker-Planck equation for the conditional process. By using the results of appendix B, the conditional probability density function of the velocity is

$$p(v|\sigma) = \frac{1}{\sqrt{2\pi}A_v\sigma} \exp\left(-\frac{v^2}{2A_v{}^2\sigma^2}\right) \tag{8}$$

Thus, the velocity is a Gaussian random variable. The standard deviation factor of the velocity A_v is independent of the amplitude parameter. In the present case the velocity is not affected by the nonlinearity in the stiffness force.

The exceedance expression (the expected frequency of positive slope crossings of a given level) is determined from the joint probability density function of the displacement and velocity (refs. 18 and 19) as

$$N(x|\sigma) = \int_0^\infty v\, p(x,v|\sigma)\, dv \tag{9}$$

The relation between the exceedance expression and the probability density function of the displacement is obtained by substituting equations (3) and (8) into equation (9) to obtain

$$N(x|\sigma) = \frac{A_v\sigma}{\sqrt{2\pi}}\, p(x|\sigma) \tag{10}$$

In this case the exceedance expression has the same functional form as the probability density of the displacement. This property is a consequence of the independence of the displacement and velocity for the conditional process (eq. (3)).

The preceding analysis shows that it is possible to develop the exact solution for the probability density function of the system response in the case of the bilinear spring system. Approximate methods of solution are also examined, since the approximate methods can be applied to a wider class of nonlinear dynamic systems. For the present system the results of approximate solution methods can be compared with the exact solution; thus, the validity and limitations of the approximation methods are indicated.

Dempster-Roger approximation.- One approximate solution method was developed by Dempster and Roger (ref. 3). The Dempster-Roger approximation, which was developed from the results of analog computer studies of nonlinear systems, specifically considers the variation of the standard deviation of the displacement with the amplitude parameter σ. For small values of the amplitude parameter the standard deviation of the displacement is determined by the inner region of the stiffness curve. For large values the standard deviation is determined by the outer region of the stiffness curve. Accordingly, the variation of the standard deviation of the displacement with the amplitude parameter is approximated with a bilinear relation

$$\sigma_{xc,dr} = \begin{cases} A_{x1}\sigma & (0 \leqq \sigma \leqq \sigma_b) \\ A_{x2}\sigma + (A_{x1} - A_{x2})\sigma_b & (\sigma_b < \sigma) \end{cases} \tag{11}$$

The standard deviation of the displacement can be determined within the accuracy of the Dempster-Roger approximation if the value of the amplitude parameter breakpoint σ_b can be determined. Since the exact relation is known in this case, the value of this breakpoint can be determined. By using the exact relation

for the standard deviation of the displacement (eqs. (6) and (7)), and developing the asymptotic expansion for large values of the amplitude parameter, the relation for the amplitude parameter breakpoint is

$$\sigma_b = \frac{1}{\sqrt{2\pi}} \frac{x_b}{A_{x1}}(\gamma + 1) \tag{12}$$

Based upon analog computer studies of control surface limiting effects in aircraft response problems, the following relation was used in reference 3:

$$\sigma_b = \frac{1}{\sqrt{2}} \frac{x_b}{A_{x1}} \tag{13}$$

This expression has the same functional relation between the system parameters as equation (12), but omits the dependence on the response in the second linear region which appears through the parameter γ. For the cases considered in reference 3, equation (13) predicts values of the amplitude parameter breakpoint which are smaller than the values given by equation (12).

A comparison of the Dempster-Roger approximation (eqs. (11) and (12)) and the exact relation (eqs. (6) and (7)) for the standard deviation of the displacement is plotted in figure 4 in nondimensional form as a function of the amplitude parameter. The approximation gives the correct behavior in the limits of small and large values of the amplitude parameter. Figure 5 shows the corresponding comparison of the approximate and exact values of the standard deviation factor of the displacement, which is plotted in nondimensional form as a function of the amplitude parameter. (Fig. 5 also shows results from another approximate method which are discussed subsequently.) The exact relation is the same as that plotted in figure 2. Again, the Dempster-Roger approximation gives the correct behavior in the limits of small and large values of the amplitude parameter, but gives a low estimation of the standard deviation factor for intermediate values.

In the Dempster-Roger approximation the bilinearity in the displacement is replaced with a bilinearity in the amplitude parameter σ. The resulting approximate dynamic system is linear for the response to the conditional process, that is, for a given value of the amplitude parameter. The conditional response of the nonlinear system is that of a linear system whose standard deviation factor for the displacement depends on the amplitude parameter. Thus, the system response to the conditional process is Gaussian. The displacement has a zero mean value and a standard deviation given by equation (11). The probability density function of the velocity is not affected by the stiffness nonlinearity, an assumption which matches the exact solution in this case. Thus, the joint probability density function of the displacement and velocity, and the exceedance expression are known for the conditional response.

Equivalent linearization approximation.- Another method of appproximate solution for the response of nonlinear dynamic systems is the equivalent linearization technique. Discussions of this technique are given in references 10, 11, 19, and 20.

In the equivalent linearization technique the nonlinear response forces are replaced by equivalent linear relations. The differential equation of the equivalent linear system is that of the original system (eq. (1)), the nonlinear stiffness force being replaced by the equivalent linear relation

$$m\ddot{x} + \beta\dot{x} + k_e(\sigma)\, x = \sigma\, \xi(t) \tag{14}$$

The coefficients of the equivalent linear forces are determined by the condition that the variance of the difference between the nonlinear and the equivalent linear response forces be a minimum. For the present system with nonlinearity in the stiffness force only, the variance to be minimized is

$$E\left\{[f(x) - k_e(\sigma)\, x]^2 \middle| \sigma\right\} \tag{15}$$

The value of the equivalent linear stiffness coefficient $k_e(\sigma)$ which minimizes this quantity is

$$k_e(\sigma) = \frac{E[x\ f(x)|\sigma]}{E\left[x^2|\sigma\right]} \tag{16}$$

The indicated statistical averages are determined from the conditional probability density function of the response of the equivalent linear system. The specific relations for the required statistical averages are

$$E[x\ f(x)|\sigma] = \int_{-\infty}^{\infty} x\ f(x)\ p_e(x|\sigma)\ dx \tag{17}$$

$$E\left[x^2|\sigma\right] = \sigma_{xc,e}^2 = A_{xe}^2(\sigma)\ \sigma^2 \tag{18}$$

The displacement and velocity of the response of the equivalent linear system to the Gaussian conditional process are both Gaussian and are independent. By using the conditional variance (eq. (18)), the probability density function of the displacement is

$$p_e(x|\sigma) = \frac{1}{\sqrt{2\pi}A_{xe}\sigma}\ e^{-x^2/2A_{xe}^2\sigma^2} \tag{19}$$

Since the standard deviation of the velocity is not affected by the nonlinearity of the stiffness force in this case, the associated probability density function is the same as the exact relation (eq. (8)).

The equivalent linear stiffness coefficient and the standard deviation factor of the displacement are related in a simple manner for the linear system of equation (14) with white noise excitation:

$$\frac{A_{xe}^2(\sigma)}{A_{x1}^2} = \frac{k_1}{k_e(\sigma)} \tag{20}$$

Combining equations (16) to (18) with equation (19) and using the relation for the bilinear stiffness force equation (2) yields the relation for the equivalent linear stiffness coefficient as

$$k_e(\sigma) = k_1 \operatorname{erf}\left(\frac{x_b}{\sqrt{2}A_{xe}\sigma}\right) + k_2 \operatorname{erfc}\left(\frac{x_b}{\sqrt{2}A_{xe}\sigma}\right) \tag{21}$$

The equivalent linear stiffness coefficient and the associated standard deviation factor of the displacement are determined as a function of the amplitude parameter by combining equations (20) and (21). The equations are solved numerically by iteration. The equivalent linear stiffness coefficient is plotted in figure 6 in nondimensional form as a function of the amplitude parameter for the value of the parameter γ used in figures 4 and 5. The value of the equivalent linear coefficient approaches the values of the stiffness coefficients of the inner and outer regions of the stiffness force relation in the limits of small and large values of the amplitude parameter, respectively.

The standard deviation factor of the displacement of the equivalent linear system is plotted in figure 5 in nondimensional form as a function of the amplitude parameter. The standard deviation factor corresponds to the values of the equivalent linear stiffness coefficient shown in figure 6; the two quantities are related by equation (20). From figure 5 it is seen that the equivalent linearization method gives results which are very close to the exact results for the standard deviation factor of the displacement.

Amplitude Modulated Process

The solution for the response of the bilinear spring system to excitation by the amplitude modulated random process is devel-

oped in this section. The formulation and the analysis of the response of dynamic systems to the amplitude modulated random process are discussed in appendix A. The response of a dynamic system to the amplitude modulated process is developed from the response to the Gaussian conditional process by introducing the random variation of the amplitude parameter. The probability density function of the amplitude parameter is specified to have a Gaussian related form:

$$p(\sigma) = \begin{cases} 0 & (\sigma < 0) \\ \sqrt{\frac{2}{\pi}} \frac{1}{b} e^{-\sigma^2/2b^2} & (0 \leq \sigma) \end{cases} \tag{22}$$

The probability density function of the displacement of the bilinear spring system is determined from the conditional probability density function of the displacement and the probability density function of the amplitude parameter

$$p(x) = \int_0^\infty p(x|\sigma)\, p(\sigma)\, d\sigma \tag{23}$$

Exact solution.- The exact solution for the response of the bilinear spring system to the amplitude modulated process is obtained by combining the exact solution of the response to the conditional process with the random variation of the amplitude parameter. Although the resulting expressions are thus identified as the exact solution, the development of the expressions for the response to the amplitude modulated process requires the use of the quasi-steady approximation, that is, the dynamic effects of the slowly varying amplitude process are omitted. In this sense the resulting expressions are approximate, as discussed in appendix A.

The exact expression for the conditional probability density function of the displacement is equation (4). By combining equations (4) and (22), the probability density function of the displacement response to the amplitude modulated random process is obtained from equation (23). The resulting probability density function of the displacement cannot be expressed in analytical form since the required integration over the σ variable is intractable.

The variance of the displacement is obtained from the conditional variance by using the probability density function (eq. (23))

$$E\left[x^2\right] = \sigma_x^2 = \int_0^\infty E\left[x^2|\sigma\right] p(\sigma)\, d\sigma = \int_0^\infty A_x^2(\sigma)\, \sigma^2\, p(\sigma)\, d\sigma \qquad (24)$$

The last expression introduces the standard deviation factor by using equation (6). The required integration over the amplitude parameter σ is intractable and must be evaluated numerically. The exact values of the standard deviation of the displacement are shown in figure 7 in nondimensional form as a function of the standard deviation b of the amplitude process. (Fig. 7 also shows two approximate results which are discussed subsequently.) The results in figure 7 for the response to the amplitude modulated process correspond to those in figure 5 for the conditional process. The standard deviation of the displacement approaches that of the two limiting linear systems of the stiffness force relation for the limiting values of both the standard deviation of the amplitude process and the breakpoint of the stiffness force relation.

The probability density function of the velocity of the system response to the amplitude modulated process is determined from that for the response to the conditional process as

$$p(v) = \int_0^\infty p(v|\sigma)\, p(\sigma)\, d\sigma \tag{25}$$

For the conditional process the velocity of the system response is Gaussian (eq. (8)). Thus, the velocity is itself an amplitude modulated variable for the response of a linear system

$$p(v) = \frac{1}{\pi A_v b} K_0\left(\frac{|v|}{A_v b}\right) \tag{26}$$

The quantity K_0 is a modified Bessel function of zero order (ref. 17).

The exceedance expression for the displacement response to the amplitude modulated process is obtained from that for the conditional process

$$N(x) = \int_0^\infty N(x|\sigma)\, p(\sigma)\, d\sigma \tag{27}$$

This equation follows from the relation between the exceedance expression and the joint probability density of the displacement and velocity, by using the relation for that probability density function for the amplitude modulated process which is similar to equation (23). By using the exceedance expression (eq. (10)) for the conditional process and equation (27), the relation for the exceedance expression of the displacement response to the amplitude modulated process is

$$N(x) = \frac{A_v}{\sqrt{2\pi}} \int_0^\infty p(x|\sigma)\, p(\sigma)\, \sigma\, d\sigma \tag{28}$$

The two probability density functions are given in equations (4) and (22). The required integration over the σ variable is

intractable and must be evaluated numerically. The resulting exceedance expression for the exact solution of the system displacement is plotted in figure 8 in nondimensional form as a function of the displacement level. The system parameters are the same as those used in figure 7. The exact exceedance expression shows the predominantly exponential behavior of the amplitude modulated process. For comparison, the exceedance expressions for the limiting linear systems corresponding to the two regions of the bilinear stiffness force relation (eq. (A12)) are also plotted. The exact exceedance expression shows the dominant influence of the inner and outer regions of the bilinear stiffness in the limits of small and large values of the nondimensional response level, respectively. For large response levels the exact exceedance expression parallels that of the linear system of the outer region of the bilinear stiffness force relation, but is significantly lower in value.

Dempster-Roger approximation.- The Dempster-Roger approximation specifically considers the variation of the amplitude parameter in the conditional process. The corresponding variance of the response to the amplitude modulated process follows from the conditional standard deviation (eq. (11)). The required integration over the amplitude parameter in equation (24) can be evaluated because of the simple relation which is assumed for the conditional standard deviation

$$\sigma_{x,dr}^2 = A_{x1}^2 b^2 \left\{ \operatorname{erf} \zeta + \left[\gamma^2 + 2\zeta^2(\gamma - 1)^2 \right] \operatorname{erfc} \zeta - \frac{2}{\sqrt{\pi}} \zeta(\gamma - 1)^2 e^{-\zeta^2} \right\} \tag{29}$$

where

$$\zeta = \frac{\sigma_b}{\sqrt{2}b} = \frac{\gamma + 1}{2\sqrt{\pi}} \frac{x_b}{A_{x1} b}$$

This approximation for the standard deviation of the displacement is plotted in figure 7 in nondimensional form as a function of the standard deviation b of the amplitude random process. The approximation shows the correct qualitative properties, particularly the dominant influence of the inner and outer regions of the bilinear stiffness force in the limiting cases. However, for the intermediate cases the Dempster-Roger approximation predicts values below the exact values of the standard deviation of the displacement.

Equivalent linearization approximation.- The variance of the displacement response to the amplitude modulated process for the equivalent linearization technique is obtained from equation (24) by using the approximation for the conditional variance (eqs. (20) and (21)). The required integration on the amplitude parameter is done numerically. The resulting standard deviation of the displacement for the equivalent linearization technique is plotted in figure 7 in nondimensional form as a function of the standard deviation of the amplitude process. The standard deviation obtained by the equivalent linearization technique closely matches the exact results for all values of the standard deviation of the amplitude process.

The corresponding exceedance expression is determined from equation (28) by using the conditional probability density function of the displacement for the equivalent linear system (eqs. (19) to (21)). The required integration is done numerically. The resulting exceedance expression is plotted in figure 9 in nondimensional form as a function of the displacement level, together with the corresponding exact expression from figure 8. The exceedance expression for the equivalent linearization technique closely matches the exact expression at low response levels. At higher response levels the approximate exceedance expression underestimates the system displacement. The underestimation of the exact exceedance expression is consistent with the underestimation of the exact values of the standard deviation (fig. 7). Also, a comparison of figures 7 and 9 shows that the equivalent

linearization technique gives a better estimate of the standard deviation than of the exceedance expression of the system displacement. This is a general property since the technique does not account for the non-Gaussian aspect of the response to the conditional process.

The present development applies the equivalent linearization technique to the conditional process. Thus, an equivalent linear stiffness force is determined for each value of the amplitude parameter. The equivalent linearization technique can be applied in an alternate manner, which is directly to the amplitude modulated process. In this alternate approach the nonlinear system is replaced by a single linear system without considering the conditional process. By using the equivalent linearization condition, the minimization of the variance of equation (15), the expression for the stiffness coefficient of the single equivalent linear system is

$$k_{e,am} = \frac{E[x\ f(x)]}{E[x^2]} \tag{30}$$

In this case the response of the equivalent linear system is itself an amplitude modulated process for the response of a linear system

$$p_{e,am}(x) = \frac{1}{\pi A_{xe,am^b}} K_0\left(\frac{|x|}{A_{xe,am^b}}\right) \tag{31}$$

$$N_{e,am}(x) = N_0 \exp\left(\frac{-|x|}{A_{xe,am^b}}\right) \tag{32}$$

The statistical averages in equation (30) are based on the probability density function of the single equivalent linear system (eq. (31)). The equivalent linear stiffness coefficient (eq. (30)) is a function of the standard deviation b of the amplitude process.

The resulting variance and exceedance expression are thus determined for a given value b. The exceedance expression is a single exponential function (eq. (32)). This relation would appear as a linear function on the coordinates of figure 9. Comparison of this approximate relation with the exact exceedance expression shows that a single exponential function is a poor approximation for the entire range of the displacement levels. Thus, the application of the equivalent linearization technique directly to the amplitude modulated process gives a significantly poorer approximation than the indirect application through the conditional process discussed previously.

Summary and Discussion of Response of Symmetric Bilinear Spring System

The response of the dynamic system with a symmetric bilinear spring to excitation by the amplitude modulated white noise process is determined by Fokker-Planck equation. The solution procedure involves determining the response to the Gaussian conditional process and then introducing the random variation of the amplitude process. The probability density functions of both the displacement and velocity, the associated variances, and exceedance expression are determined. Two approximate analytical techniques are applied to the same system. The accuracies of these techniques are assessed by comparison with the exact solution.

The approximation techniques introduce the system nonlinearity through the standard deviation factor A of the displacement response to the conditional process. The dynamic system is considered to be linear for any given value of the amplitude parameter σ of the excitation conditional process. The critical point of the approximation techniques is the determination of a functional relation between the standard deviation factor and the amplitude parameter. The Dempster-Roger approximation specifies a bilinear relation between these two quantities. The limitation of this method is the lack of any general procedure for finding the

specific form of this bilinear relation. In the bilinear spring system this relation can be determined since the exact solution for the response is known. In this case the Dempster-Roger approximation gives the correct qualitative properties of the response, but generally gives values of the displacement response which are appreciably below the exact ones. The equivalent linearization technique replaces the nonlinear system with an equivalent linear one whose properties are determined by the technique. For the bilinear spring system the equivalent linearization technique, which is applied to the conditional process, gives a good approximation to the exact values of the system response to the amplitude modulated process.

The preceding development considers the response of a symmetric nonlinear system. As a result of the symmetry property, all the odd order moments are zero. The case of a nonsymmetric bilinear spring system is considered briefly in appendix C.

RESPONSE OF AUGMENTED AIRCRAFT SYSTEM

A method for the analysis of nonlinear effects in aircraft dynamic response to atmospheric turbulence is developed in the present section. The method is the combination of the application of the equivalent linearization technique for the analysis of nonlinear dynamic systems with the development of the amplitude modulated process used to model atmospheric turbulence. The equivalent linearization technique is applied to replace the nonlinear dynamic system with an equivalent linear system. The response of the resulting dynamic system to a Gaussian conditional process with a given amplitude parameter is developed by the methods of linear system theory. The response of the system to the amplitude modulated process is then developed by introducing the random variation of the amplitude parameter. The method of analysis is applied to one example of the longitudinal motion of an aircraft with an autopilot operating in a pitch-hold mode. The nonlinear

element is the limiting of the control surface displacement which is commanded by the autopilot.

Equations of Augmented Aircraft

An aircraft which is representative of the class of small corporate jet transports is considered. The present combination of aircraft and stability augmentation system was considered in detail in reference 21. The longitudinal equations of motion and the associated control law for the linear aircraft system are summarized in appendix D. The linear equations of motion are basically the classical equations of the dynamic stability of a rigid aircraft (ref. 22) with the addition of the forces due to the turbulence field. Quasi-steady aerodynamics are used except for the inclusion of a wing-tail convective time lag. The control system operates in a pitch-hold mode by controlling the elevator deflection.

A block diagram of the basic aircraft response system, including the nonlinear effect of the control surface displacement limiting, is shown in figure 10. The aircraft dynamics consist of the linear response of the basic aircraft motion. The feedback loop consists of four elements. The first element is the sensor which measures the pitch and pitch rate of the aircraft motion. The second element accounts for the dynamics of the servosystem which drives the elevator deflection. These two elements form the control law of the autopilot system. The third element is the displacement limiting of the elevator deflection. The fourth element generates the aerodynamic forces due to the physical motion of the control surface. Except for the displacement limiting of the elevator, the aircraft system is linear.

The control law specifies the elevator deflection which is commanded by the control system. Since the autopilot is designed to reduce aircraft pitching motion, the control system consists of gains on the pitch and the pitch rate motions and includes a term accounting for the dynamics of the elevator servosystem

$$t_{ch}\dot{\delta}_c + \delta_c = K_\theta \theta + K_{\dot{\theta}}\dot{\theta} \tag{33}$$

The aerodynamic forces which are generated by the physical deflection of the elevator are given by linear aerodynamic theory The relation for the aircraft moment due to the physical elevator deflection is

$$M_\delta = q_d S\bar{c} C_{m_\delta} \delta_p \tag{34}$$

The main quantity of interest is the aerodynamic moment coefficient for the elevator deflection C_{m_δ}. The aerodynamic lift due to the physical elevator deflection is given by a similar relation.

Both the control law and the elevator aerodynamic relations are linear. There are two variables associated with the elevator action: the commanded deflection from the control law and the physical deflection which generates the aerodynamic forces. These two quantities are related in a nonlinear manner because of the deflection limiting

$$\delta_p = h(\delta_c) = \begin{cases} -\delta_{pl} & (\delta_c < -\delta_{pl}) \\ \delta_c & (-\delta_{pl} \leqq \delta_c \leqq \delta_{pl}) \\ \delta_{pl} & (\delta_{pl} < \delta_c) \end{cases} \tag{35}$$

This relation is plotted in figure 11. For commanded deflections within the symmetric limit values, the two quantities are equal. If the commanded deflection is outside the limit values, then the physical deflection is equal to the limit values. Since the aerodynamic forces are proportional to the physical elevator deflection (eq. (34)), the relation plotted in figure 11 is also the relation between the aerodynamic forces and the commanded elevator deflection.

The preceding relations are combined with those for the basic aircraft dynamics to give the nonlinear equations of motion for the augmented aircraft. The system of equations has three coordinates of basic aircraft motion: perturbations in forward speed, angle of attack, and pitch angle. The system has two coordinates of elevator action: commanded and physical elevator deflections. The system has three equations for the basic aircraft motion which include the aerodynamic forces due to the physical elevator deflection. The system has two equations related to the control system action: the control law (eq. (33)) and the nonlinear relation between the two elevator deflection variables (eq. (35)).

Application of Equivalent Linearization Technique

The results of the previous application of this method to the bilinear spring system give reasonable confidence that the method will give a good estimation of the response of the present aircraft system. The confidence in the application of the method is based upon the similarity in the mathematical properties of the two dynamic systems. This point is discussed before applying the method to the aircraft system.

Similarity of bilinear spring and aircraft systems.- The differential equation of the bilinear spring system (eq. (1)) is rewritten in a form which is similar to that of the augmented aircraft

$$m\ddot{x} + \beta\dot{x} + k_2 x = -(k_1 - k_2)\, h(x) + \sigma\, \xi(t) \qquad (36)$$

The response stiffness force has been split into two terms: the basic linear term of the outer linear region and the nonlinear term which is now a control system feedback loop. The nonlinear displacement function $h(x)$ is that of equation (35) and figure 11 with appropriate change in the notation. A block diagram of the resulting system is shown in figure 12. (The quantity s is the Laplace transform variable.) In the block diagram the basic

linear dynamic system is excited by a Gaussian white noise process with amplitude parameter σ and by the nonlinear feedback term.

The block diagram of the augmented aircraft system is shown in figure 13. This diagram gives specific relations for the system elements shown in figure 10. The relation for the aircraft dynamics gives the pitch response to the aerodynamic force inputs. In developing this relation, whose specific form is given in appendix D, the secondary effects of the forward speed perturbation and wing-tail convective time lag have been omitted. The feedback loop has four elements which correspond to those of figure 10.

Comparison of figures 12 and 13 reveals the similarity of the bilinear spring and the aircraft systems. The dynamic properties of the two basic systems have the same functional form, that is, a single oscillatory mode. Both feedback systems have an amplitude-limited displacement which generates a restoring force for the basic dynamic system. There are, however, several differences in the feedback loops. The autopilot system has an additional gain on the pitch rate which has a minor effect on the pitch response. The other differences in the feedback loop involve dynamic effects in the servosystem and the elevator aerodynamics. For the parameters of the present problem these dynamic effects are negligible at the frequency of the dominant short-period mode. The remaining difference between the two dynamic systems is the form of the two excitation functions. Since the response of the two dynamic systems is dominated by a single oscillatory mode, the variances of the response are closely related to the values of the power spectral density function of the excitation process at the frequency of that mode. However, the nonlinear effects introduce a change in the frequency of the oscillatory mode. Thus, the variance of the response is influenced by the frequency dependence of the power spectral density function of the excitation process, which for atmospheric turbulence can be considerable. Since the power spectral density function of a white noise process has a constant value, this effect is missing in the response of the bilinear spring system. Thus, there are significant differences in some of

the properties of the response of the bilinear spring and the aircraft systems as discussed subsequently.

Control effectiveness factor.- In the application of the equivalent linearization technique the nonlinear element is replaced by an equivalent linear element whose properties are defined so that the variance of the difference between the responses of the linear and nonlinear elements is a minimum. In the present case this condition is applied to the nonlinear relation between the two elevator deflection coordinates (eq. (35)). The equivalent linear relation for the two elevator deflection coordinates is

$$\delta_p = h_e \delta_c \tag{37}$$

The quantity h_e is the control effectiveness factor (or simply the effectiveness factor) of the equivalent linear system, which accounts for the elevator deflection limiting. The variance of the difference between the nonlinear and the equivalent linear relations for the physical elevator deflection is

$$E\left\{\left[h(\delta_c) - h_e \delta_c\right]^2 \middle| \sigma\right\} \tag{38}$$

The control effectiveness factor is defined to minimize this variance

$$h_e = \frac{E\left[h(\delta_c)\ \delta_c \middle| \sigma\right]}{E\left[\delta_c{}^2 \middle| \sigma\right]} \tag{39}$$

The indicated statistical averages are based on the response of the equivalent linear system to the conditional process. The system input and the system response are both Gaussian conditional processes. The probability density function of the commanded elevator deflection is

$$p(\delta_c|\sigma) = \frac{1}{\sqrt{2\pi}A_{\delta_c e}\sigma} \exp\left(\frac{-\delta_c^2}{2A_{\delta_c e}^2\sigma^2}\right) \tag{40}$$

The control effectiveness factor is determined by equation (39), by using the functional relation for the physical elevator deflection (eq. (35)) and by using the probability density function of the commanded elevator deflection of the linearized system (eq. (40))

$$h_e = \text{erf}\left(\frac{\delta_{pl}}{\sqrt{2}A_{\delta_c e}\sigma}\right) \tag{41}$$

(This relation is a special case of eq. (21).) The control effectiveness factor has a maximum value of one and a minimum value of zero and approaches these values in the limits of small and large values of the amplitude parameter, respectively. It is noted that the minimum variance condition on the difference of the elevator deflections (eq. (38)) is also a minimum variance condition on the differences of the associated elevator aerodynamic forces (eq. (34)) since the quantities are proportional in the present case.

Once a linear relation between the two elevator deflection coordinates (eq. (37)) is obtained, one of these coordinates can be eliminated from the equations of the dynamic system. One approach is the elimination of the physical elevator deflection. The resulting control law and aerodynamic force relation are

$$t_{ch}\dot{\delta}_c + \delta_c = K_\theta\theta + K_{\dot{\theta}}\dot{\theta} \tag{42a}$$

$$M_\delta = q_d S\bar{c} C_{m_\delta} h_e \delta_c \tag{42b}$$

The equations for the resulting equivalent linear system are the same as for the original linear system of appendix D except that the effectiveness factor is introduced into the aerodynamic derivatives of the elevator deflection. The elevator deflection

coordinate of the linear system becomes the commanded elevator deflection. The solution of the linear equations gives the standard deviation factor of the commanded elevator deflection which is required for the basic relation (eq. (41)). An alternate but equivalent approach is the elimination of the commanded elevator deflection. The resulting control law and aerodynamic force relations are

$$t_{ch}\dot{\delta}_p + \delta_p = h_e K_\theta \theta + h_e K_{\dot{\theta}} \dot{\theta} \tag{43a}$$

$$M_\delta = q_d S \bar{c} C_{m_\delta} \delta_p \tag{43b}$$

In this form the effectiveness factor is introduced into the two control system gains. The elevator deflection coordinate of the linear system becomes the physical elevator deflection. The solution of the linear equations gives the standard deviation factor of the physical elevator deflection. This relation must be transformed by equation (37) to give the standard deviation factor of the commanded deflection which is required for the basic relation (eq. (41)).

The preceding development gives the essential relations for applying the equivalent linearization technique to the nonlinear control-surface limiting effect in aircraft response. The application follows this procedure. First, the equivalent linearization technique is applied to the nonlinear system and thus introduces the effectiveness factor into the equations of the dynamic system. Second, the resulting linear equations of motion are solved for the response to the Gaussian conditional process, for a series of values of the effectiveness factor between zero and one. This procedure assumes that the equivalent linear system is stable for all values of the effectiveness factor. The solution gives the standard deviation factors of the conditional response, which are functions of the effectiveness factor but not of the amplitude parameter σ. Third, the relation between the standard deviation factor of the commanded elevator deflection coordinate and the

effectiveness factor is combined with the basic relation (eq. (41)). By eliminating the elevator quantity, the effectiveness factor becomes a known function of the amplitude parameter. By using this relation, the standard deviation factors of all response quantities are known functions of the amplitude parameter. Finally, the system response to the amplitude modulated process is developed from the conditional response by introducing the random variation of the amplitude parameter.

Development of Response of Augmented Aircraft System

The procedure for implementing the equivalent linearization technique is applied to the example of control surface limiting in aircraft response to atmospheric turbulence. The example uses the equations of motion and the data of reference 21, which are discussed in appendix D. The effectiveness factor is introduced into the linear equations of motions in the terms for the aerodynamic lift and moment due to the elevator deflection. The equations are solved for the standard deviation factors of the aircraft response, for a series of values of the effectiveness factor between zero and one. The resulting standard deviation factors of both the commanded and the physical elevator deflection coordinates are plotted in figure 14 in normalized form as functions of the effectiveness factor. They are normalized by their maximum values, which are equal to the standard deviation factor for the elevator deflection of the fully effective control system. As the deflection limiting becomes significant, the value of the effectiveness factor decreases from the value of one down to zero and the standard deviation factors of both elevator deflection coordinates decrease from their maximum values down to zero. The difference between the standard deviation factors of the commanded and the physical elevator deflections is a direct result of the elevator deflection limiting. The decrease of the standard deviation factor of the commanded elevator deflection from its maximum value is the effect of the elevator deflection limiting upon the

aircraft dynamic system. By using the results of figure 14 and the basic relation (eq. (41)), the relation between the effectiveness factor and the amplitude parameter is determined by eliminating the standard deviation factor of the commanded elevator deflection. The resulting relation is plotted in figure 15 in nondimensional form. By using this relation, the standard deviation factors of all response quantities, which are known functions of the effectiveness factor from the solution of the equations of motion, become known functions of the amplitude parameter. For the elevator deflection coordinates, this relation is obtained by combining results of figures 14 and 15; thus, figure 16 shows the relations in nondimensional form as functions of the amplitude parameter. These relations show the transition from the full activity of the elevator to the reduced activity as either the amplitude parameter of the input random process is increased or the elevator deflection limit is decreased.

The standard deviations of the two elevator deflection coordinates for the response to the amplitude modulated random process are obtained from the conditional response by introducing the random variation of the amplitude parameter (eqs. (22) and (23)). The resulting standard deviations for the response to the amplitude modulated process are plotted in figure 17 in nondimensional form as functions of the standard deviation of the amplitude modulated process. The response shows the transition from the full to the reduced activity of the control system as either the standard deviation of the amplitude modulated process is increased or the elevator deflection limit is decreased.

The pitch response of the aircraft is examined next. This quantity particularly shows the effect of the elevator deflection limiting since the control system is designed to reduce the pitch angle changes due to the atmospheric turbulence. The relation between the standard deviation factor of the pitch angle and the effectiveness factor is shown in figure 18 in nondimensional form. This relation is obtained from the solution of the equivalent linear form of the dynamic equations of the augmented aircraft

system. By combining the relations of figures 15 and 18, the standard deviation factor of the pitch coordinate is a known function of the amplitude parameter. The resulting relation is shown in figure 19 in nondimensional form. The standard deviation factor of the pitch coordinate shows the transition from its minimum value with full operation of the control system to the increased values as the control system effectiveness is reduced because of the elevator deflection limiting with increased values of the amplitude parameter. The pitch response to the amplitude modulated process is obtained by introducing the random variation of the amplitude parameter. The resulting relation for the standard deviation of the pitch coordinate is plotted in figure 20 in nondimensional form as a function of the standard deviation of the amplitude modulated process. From the result of figure 20, the pitch response approaches that of the fully augmented airplane as the effect of the elevator deflection limiting vanishes in the limit either of small values of the turbulence intensity or of large values of the elevator deflection limits. In the opposite limits the pitch response approaches that of the unaugmented aircraft.

The pitch rate is now examined. By following the steps of the procedure, the relation between the standard deviation factor of the pitch rate and the amplitude parameter is determined. The resulting relation is plotted in figure 21 in nondimensional form. The dependence upon the amplitude parameter is weaker for the pitch rate than for the pitch itself (fig. 19). However, the standard deviation factor of the response rate is not independent of the amplitude parameter as it is in the case with the bilinear spring system. This difference between the responses of the two dynamic systems is primarily the result of the differences between the spectral functions of the excitation processes of the two systems.

The exceedance expression for the pitch response is developed from the previous results. Since the conditional probability density functions of the pitch and pitch rate are both Gaussian for the response of the equivalent linear system, the exceedance

expression for the conditional process is known. The exceedance expression for the response to the amplitude modulated process is determined from the conditional expression and equation (27) as

$$N(\theta) = \frac{1}{2\pi} \int_0^\infty \frac{\dot{A}_{\theta e}(\sigma)}{A_{\theta e}(\sigma)} \cdot \exp\left[\frac{-\theta^2}{2A_{\theta e}^2(\sigma)\sigma^2}\right] p(\sigma)\, d\sigma \qquad (44)$$

The two standard deviation factors are known functions of the amplitude parameter (figs. 19 and 21). The required integration is performed numerically. The resulting exceedance expression is shown in figure 22 in nondimensional form as a function of the pitch response level for two values of the elevator displacement limit. The exceedance expression shows the increased pitch response of the aircraft as the value of the elevator deflection limit is decreased. The exceedance expressions for the aircraft pitch response with the autopilot on and with the autopilot off are also shown. The exceedance expression generally follows that of the augmented aircraft at low response levels. At the higher response levels the exceedance expressions begin to parallel that of the unaugmented aircraft, but are significantly lower in value.

Discussion

The present method of the combination of the equivalent linearization technique with the development of the amplitude modulated random process can be applied to more general forms of nonlinearity in dynamic systems. A variety of nonlinear effects can be important in aircraft response to atmospheric turbulence, such as the inherent nonlinearities in the aircraft equations of motion, aerodynamic effects, pilot action, and several aspects of control system action. The present example uses a simple form of the equivalent linearization technique which replaces a static nonlinearity by a single linear gain. Many other forms of the technique have been developed which allow the application of the method to a wide variety of nonlinear dynamic systems. The

formulation and application of the equivalent linearization technique to combinations of static nonlinear elements are given in references 11 and 23. Several forms of the technique which consider the dynamic properties of the nonlinear elements are reviewed in references 12 and 20. These alternate forms of the equivalent linearization technique can be directly applied to the present method of analysis since the application of the technique is separate from the development of the amplitude modulated process.

The present development can be extended to include other functional forms of the probability density of the amplitude parameter. In the quasi-steady approximation, there is no restriction on the form of the random variation of the amplitude parameter since the dynamic effects of that random variation are omitted. The present development has considered a Gaussian related form of the probability density function of the amplitude parameter. In aeronautical applications a modification of this form is used for atmospheric turbulence based on the concept of multiple types of turbulence (ref. 13). The probability density function of the amplitude parameter is formed by the sum of two (or more) Gaussian related functions

$$p(\sigma) = \begin{cases} 0 & (\sigma < 0) \\ \sqrt{\frac{2}{\pi}} \sum_{i=1}^{2} P_i \frac{1}{b_i} e^{-\sigma^2/2b_i^2} & (\sigma \geq 0) \end{cases} \tag{45}$$

The quantities P_i and b_i are the probability and intensity parameters of atmospheric turbulence, respectively. The response of the augmented aircraft system to an amplitude modulated process with this form for the random variation of the amplitude parameter can be developed from the previous results. For example, the exceedance expression is found from the general relation (eq. (27)). By using the nondimensional form of the previous results, the

new exceedance expression for the aircraft pitch response is

$$N(\theta) = \sum_{i=1}^{2} P_i N\left(\frac{\theta}{A_{\theta 1} b_i}\right) \tag{46}$$

The results for other response quantities can be developed in the same manner.

The present analysis is restricted to symmetric nonlinear systems; thus, the response has zero mean values. This restriction applies to cases of aircraft response where the control surface limiting is symmetric. If the limiting is not symmetric, then a modified form of the analysis must be used. The case of a nonsymmetric bilinear spring system is examined in appendix C and includes the analysis of the response to both the conditional and the amplitude modulated processes. The primary new feature of the response is the possibility of a nonzero mean value. Nonsymmetric control surface limiting effects can be important in aircraft dynamic response, for example, in an aircraft with a pitch damper in response to the vertical component of the turbulence. By using the nonsymmetric bilinear spring system as an analogy, the mean position of the control surface will change with significant limiting in extreme turbulence, that is, the aircraft retrims itself.

CONCLUDING REMARKS

A method is developed for the analysis of nonlinearities in aircraft dynamic response to random atmospheric turbulence. The method is a combination of the equivalent linearization technique for the analysis of nonlinear dynamic systems and the development of the amplitude modulated random process. The equivalent linearization technique is used to replace a nonlinear element by an equivalent linear one. The response of the equivalent linear system to a Gaussian random process is developed as a function of a linearization parameter. By using the equivalent linearization

technique, the linearization parameter is related to the amplitude parameter of the excitation random process. Thus, the response to the Gaussian process is known as a function of the amplitude parameter. The response to the amplitude modulated process is then determined by introducing the random variation of the amplitude parameter.

The method is applied to the analysis of two nonlinear dynamic systems. The response of a spring-mass-damper system with a bilinear spring is analyzed. For this system an exact solution for the joint probability density function of the displacement and velocity is obtained by use of the Fokker-Planck equation. The system response obtained by the equivalent linearization technique shows good agreement with the exact solution for the response. The response of an aircraft with an autopilot which drives a displacement-limited control surface is also analyzed. The analytical method is used to determine the moments and the exceedance expressions of the aircraft response.

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APPENDIX A

THE AMPLITUDE MODULATED RANDOM PROCESS AND THE FOKKER-PLANCK EQUATION

The basic analytical techniques used in the present study are discussed in this appendix. The amplitude modulated random process and the Fokker-Planck equation are discussed, together with an application of the Fokker-Planck equation to the analysis of the response of dynamic systems to the amplitude modulated random process.

Amplitude Modulated Random Process

The amplitude modulated random process is formed by a local Gaussian process in combination with a slower random modulation of the standard deviation of the local process. The random process is used to model atmospheric turbulence in many aeronautical applications. The random process was originally developed to account for the properties of measured atmospheric turbulence data (refs. 13 and 14). The mathematical properties of the process have been examined and developed in reference 16 where the amplitude modulated random process was referred to as the Press model of atmospheric turbulence.

The defining relation for the amplitude modulated process is

$$z(t) = r(t)\, s(t) \tag{A1}$$

The two component processes (R and S) are specified to be independent and stationary. The R process is a rapidly varying local Gaussian process; the S process is a slowly varying amplitude process which modulates the R component. The joint probability density function of the amplitude modulated process and the amplitude component is developed from the defining product relation (ref. 24)

$$p(z,s) = \frac{1}{|s|}\, p_r(r=z/s)\; p(s) \tag{A2}$$

An associated conditional process can be developed from the two component processes. By using the definition of the conditional probability density function and equation (A2), the conditional function is

$$p(z|s) = \frac{1}{|s|}\, p_r(r=z/s) \tag{A3}$$

If the probability density functions of the R and S component processes are given, then the probability density function of the amplitude modulated process can be determined from the joint density function by using equations (A2) and (A3)

$$p(z) = \int_{-\infty}^{\infty} p(z,s)\, ds = \int_{-\infty}^{\infty} p(z|s)\; p(s)\, ds \tag{A4}$$

Since the local R process is Gaussian, the conditional process is also Gaussian by equation (A3)

$$p(z|s) = \frac{1}{\sqrt{2\pi}\, A_z |s|}\, \exp\left(\frac{-z^2}{2A_z{}^2 s^2}\right) \tag{A5}$$

This is the locally Gaussian assumption of the original development (refs. 13 and 14). The conditional process generates an associated set of conditional moments. The most important of these moments is the variance of the conditional process

$$E\left[z^2|s\right] = A_z{}^2 s^2 \tag{A6}$$

Thus the standard deviation of the conditional process is the product of the amplitude parameter s and a standard deviation factor A, which is independent of s.

The exceedance expression (the expected frequency of positive slope crossings of a given level) of the process is an important quantity in the analysis of atmospheric turbulence data and in the specification of structural fatigue and strength design criteria. It is developed from the joint probability density of the random process and its first derivative (refs. 18 and 19)

$$N(z) = \int_0^\infty \dot{z}\, p(z,\dot{z})\, d\dot{z} \tag{A7}$$

The exceedance expression for the conditional process is developed in the same manner

$$N(z|s) = \int_0^\infty \dot{z}\, p(z,\dot{z}|s)\, d\dot{z} \tag{A8}$$

The exceedance expression for the amplitude modulated process is obtained from that for the conditional process by using the relation for the joint probability density function of the random process and its first derivative which corresponds to equation (A4)

$$N(z) = \int_{-\infty}^\infty N(z|s)\, p(s)\, ds \tag{A9}$$

An associated problem is the analysis of the response of dynamic systems to the amplitude modulated random process. The problem is difficult to treat in general terms since the amplitude modulated process is not Gaussian. However, the problem can be treated approximately by using the modulation concept that the

amplitude process S is slowly varying relative to the local R component process. Under this concept, which is termed the quasi-steady approximation, the dynamic aspects of the response are due to the Gaussian R component process only. Thus, for linear systems the R component (and the conditional process) of the excitation and the response processes are both Gaussian. For nonlinear systems the problem is more complicated. The conditional process of the response is generally not Gaussian. Thus, the properties of the response can be significantly different from those of the amplitude modulated excitation process. For example, the standard deviation factor A of the nonlinear response will generally be a function of the amplitude parameter s

$$E\left[z^2|s\right] = A_z{}^2(s)\, s^2 \tag{A10}$$

The amplitude process is specified to be Gaussian, by following the original development of reference 13; that is,

$$p(s) = \frac{1}{\sqrt{2\pi}} \frac{1}{b} e^{-s^2/2b^2} \tag{A11}$$

The probabilistic structure of the amplitude modulated process is completely defined by the Gaussian distributions of the amplitude and the R component processes. For example, the exceedance expression is obtained from equations (A8) and (A9), and from use of the quasi-steady approximation

$$N(z) = N_0 e^{-|z|/A_z{}^b} \tag{A12}$$

The exponential form of the exceedance expression was the basis of the original development of the amplitude modulated process in reference 13.

APPENDIX A

Fokker-Planck Equation

The Fokker-Planck equation is a method for analysis of the response of linear and nonlinear dynamic systems to random processes. The method was originally developed in the analysis of Brownian motion (ref. 25, for example). The development of the Fokker-Planck equation and examples of its application to nonlinear dynamic systems are given in references 19 and 26. General reviews of the method and its application to the analysis of dynamic systems are presented in references 8 and 9.

The response of a dynamic system to a random process is described by the associated transition probability density function

$$p = p(\bar{y},t|\bar{y}_0,t_0) \tag{A13}$$

This quantity is the probability density function of the response at time t conditional on the given value of the response at an earlier time t_0. The quantity $\bar{y}$ is the array of independent variables of the dynamic system. The transition probability density function is determined by the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = - \frac{\partial}{\partial y_i}(\alpha_i p) + \frac{1}{2} \frac{\partial^2}{\partial y_i \partial y_j}(\beta_{ij} p) \tag{A14}$$

The convention of repeated indices implying a summation is used. The coefficients are the derivate moments, which are the limiting values of the first and second moments of the incremental response of the dynamic system

$$\left.\begin{aligned} \alpha_i &= \lim_{\delta t \to 0} \frac{1}{\delta t} \int_{-\infty}^{\infty} \delta y_i \; p(\bar{y} + \delta\bar{y}, t + \delta t | \bar{y}, t) \; d(\delta\bar{y}) \\ \beta_{ij} &= \lim_{\delta t \to 0} \frac{1}{\delta t} \int_{-\infty}^{\infty} \delta y_i \delta y_j \; p(\bar{y} + \delta\bar{y}, t + \delta t | \bar{y}, t) \; d(\delta\bar{y}) \end{aligned}\right\} \qquad (A15)$$

In the general case the differential equation for the transition probability density function has additional terms which involve higher order derivatives. However, the associated higher order derivate moments are zero and, consequently, these terms are also zero in the present application, as discussed subsequently.

Fokker-Planck Equation and the Amplitude Modulated Process

The Fokker-Planck equation is applied to the analysis of the response of dynamic systems to the random process formed by the product of two Gaussian processes (eq. (A1)). The formulation requires the development of the derivate moments for the composite dynamic system which includes the differential equations for both the dynamic system and the two component processes, R and S. In the present example the R component is a Gaussian white noise process. The amplitude component is specified to be a first-order filtering of a second Gaussian white noise process. Thus, the amplitude process S can be slowly varying relative to the local process R.

The state equation for the nonlinear dynamic system is

$$\dot{y}_i = a_0 \; f_i(\bar{y}) + g_i s \; \xi_1(t) \qquad (A16)$$

In this form the state equation consists of the array f_i of functions of the system response and the array g_i of constants which multiply the excitation random process. The excitation process is the product of two Gaussian processes, the R component being a

white noise process. The differential equation for the amplitude component process is

$$\dot{s} = -a_1 s + \sqrt{2a_1}\, b\, \xi_2(t) \tag{A17}$$

These equations are combined to give the state equation for the composite dynamic system, the state vector consisting of the response variables of the dynamic system and the amplitude process. The coefficients a_0 and a_1 are frequency scale constants which are used to identify the relative dynamic properties of the nonlinear system and the amplitude process in the subsequent discussion. The random excitation of the composite system consists of the two independent Gaussian white noise processes of equations (A16) and (A17), which have zero mean values and autocorrelations equal to Dirac delta functions

$$\left.\begin{aligned} E\left[\xi_j(t)\right] &= 0 \\ E\left[\xi_j(t_1)\ \xi_k(t_2)\right] &= \begin{cases} \delta(t_1 - t_2) & (j = k) \\ 0 & (j \neq k) \end{cases} \end{aligned}\right\} \tag{A18}$$

With this definition of the autocorrelation function of the white noise processes and with the differential equation of the amplitude process (eq. (A17)), the resulting amplitude process has the probability density function of equation (A11).

The derivate moments (eq. (A15)) are obtained from the state equation of the composite system. The derivate moments are those of a nonlinear dynamic system under white noise excitation except for those moments which correspond to the terms with the product of the amplitude and the first white noise process of equation (A16). Since these two processes are independent, the amplitude variable appears only as an amplitude factor in the derivate moments associated with this excitation term. The associated higher order derivate moments are zero as a result of both the

independence property and the Gaussian property of the white noise process. The resulting derivate moments are combined with equation (A14) to form the Fokker-Planck equation of the composite dynamic system of equations (A16) and (A17)

$$\frac{\partial p}{\partial t} = -a_0 \frac{\partial}{\partial y_i}\left[f_i(\bar{y})\ p\right] + \frac{1}{2}\ s^2 g_i g_j \frac{\partial^2 p}{\partial y_i\ \partial y_j} + a_1\left[\frac{\partial}{\partial s}(sp) + b^2 \frac{\partial^2 p}{\partial s^2}\right] \tag{A19}$$

Again, repeated indices imply a summation. The first term on the right-hand side results from the response term of the original dynamic system of equation (A16). The second term results from the excitation of the original dynamic system by the product process. This term couples the amplitude and the dynamic system variables in the Fokker-Planck equation. The last two terms give the Gaussian form of the amplitude process as a first-order filtering of white noise. The time derivative term on the left-hand side accounts for the nonstationarity of the system response. This term is droppped in the subsequent development which considers only the stationary case.

The Fokker-Planck equation gives the joint probability density function of the dynamic system variables and the amplitude process S. The equation is replaced by the corresponding equation for the conditional probability density function with the amplitude process as the conditional variable

$$p(\bar{y},s) = p_c(\bar{y}|s)\ p(s) \tag{A20}$$

By using the probability density function of the amplitude process (eq. (A11)) together with equation (A19), the Fokker-Planck equation for the conditional density function is

$$\frac{\partial}{\partial y_i}\left[f_i(\bar{y})\ p_c\right] - \frac{1}{2a_0}\ s^2 g_i g_j \frac{\partial^2 p_c}{\partial y_i\ \partial y_j} = \epsilon\left(b^2 \frac{\partial^2 p_c}{\partial s^2} - s \frac{\partial p_c}{\partial s}\right) \tag{A21}$$

where

$$\epsilon = \frac{a_1}{a_0}$$

The solution of this equation can be developed in a perturbation form which is related to the quasi-steady approximation. It is assumed that the amplitude process is slowly varying relative to the system dynamics; that is, the ratio of the associated frequency scale constants (a_1 divided by a_0) is much less than one. The conditional probability density function is accordingly written in series form with this ratio as a perturbation parameter

$$p_c(\bar{y}|s) = \sum_{n=0}^{\infty} \epsilon^n \phi_n(\bar{y},s) \tag{A22}$$

By combining the series expression with the Fokker-Planck equation (eq. (A21)), a set of equations is generated for the functions ϕ_n of the series. The equation for the first term of the series is

$$\frac{\partial}{\partial y_i}\left[f_i(\bar{y})\ \phi_0(\bar{y})\right] - \frac{1}{2a_0}\ s^2 g_i g_j\ \frac{\partial^2 \phi_0}{\partial y_i\ \partial y_j} = 0 \tag{A23}$$

This relation is the Fokker-Planck equation for the original dynamic system (eq. (A16)), the quantity s being an amplitude parameter which multiplies the excitation white noise process. The resulting probability density function is the conditional function of the system response, conditional on the value of the amplitude parameter s. The preceding development is thus the quasi-steady approximation for the analysis of the response of dynamic systems to the amplitude modulated process (eq. (A1)). The approximation is based upon the modulation concept that the time variation of the amplitude process is much slower than that

of both the R component process and all coordinates of the dynamic system. The quasi-steady form of the Fokker-Planck equation (eq. (A23)) is a limiting case of the exact equation (eq. (A21)).

The development of the quasi-steady approximation in this case depends on the relative values of the two frequency scale constants, a_0 and a_1. The definition of the constant a_0 of the nonlinear system depends upon the specific form of the system under consideration. For the bilinear spring system considered in the text, a frequency scale constant is easily defined since the nonlinear system has two limiting linear systems. In this case the frequency scale constant a_0 is the minimum value of the natural frequencies of the two limiting linear systems.

The response of dynamic systems to the amplitude modulated process can thus be analyzed by use of the quasi-steady approximation. The analysis procedure consists of two steps. First, the probability density function of the system response to a stationary Gaussian process with a constant amplitude parameter is determined by the Fokker-Planck equation. This is the conditional probability density function of the response. Second, the random variation of the amplitude process is introduced through equation (A4). For linear dynamic systems this procedure is relatively simple since the dynamic response to the Gaussian conditional process is also Gaussian. Also, the amplitude parameter appears solely as an amplitude factor, which does not change the functional form of the probability density of the response. For nonlinear dynamic systems the response to the Gaussian conditional process is generally not Gaussian. The functional form of the probability density of the response generally changes as the amplitude parameter varies. Thus, the response cannot be developed in general form as it can be for linear systems.

The amplitude component is specified to be a Gaussian process in the development of the quasi-steady approximation. This specification is required for the formulation of the exact system response including the dynamic effects of the amplitude process

(eq. (A17)). However, the dynamic properties of the amplitude process are omitted once the quasi-steady approximation has been established. The amplitude process can then be presented in a Gaussian related form which is restricted to nonnegative values

$$\sigma = |s| \tag{A24}$$

This formulation for the amplitude process is commonly used in the aeronautical literature. The original approach (ref. 13) introduced the amplitude process as a scale factor in the standard deviation of the conditional process by use of equation (A5). Under the quasi-steady approximation the formulations of the amplitude modulated process in terms of the σ and the s variables are equivalent. The properties of the two processes are related by the transformation of equation (A24). The probability density function of the modified form of the amplitude process is

$$p(\sigma) = \begin{cases} 0 & (\sigma < 0) \\ \sqrt{\frac{2}{\pi}} \frac{1}{b} e^{-\sigma^2/2b^2} & (0 \leq \sigma) \end{cases} \tag{A25}$$

The local Gaussian process (white noise in this case) and the associated system response always occur in combination with the amplitude parameter (either $|s|$ or σ). This relationship follows from the form of the system excitation function (eq. (A16)). In aeronautical applications an arbitrary scale factor between the local process and the amplitude parameter is established by defining the local process to have unit variance (ref. 7). In the present case this approach cannot be used since the variance of the local process is not defined. Thus, the standard deviation factors of the response quantities are devoid of their usual meaning: the ratio of the standard deviations of the (conditional) response and excitation processes. However, the notational separation of the standard deviation of the response into

the product of a standard deviation factor and an amplitude parameter (eqs. (A6) and (A10), for example) is retained in the white noise case in order to correspond to the usual notation of the aeronautical literature.

APPENDIX B

APPLICATION OF FOKKER-PLANCK EQUATION TO SYMMETRIC BILINEAR SPRING SYSTEM

The application of the Fokker-Planck equation to the analysis of the response of the dynamic system with a symmetric bilinear spring is developed in this appendix. This system is a special case of a general class of systems discussed in references 19 and 26.

A single-degree-of-freedom spring-mass-damper system with a nonlinear stiffness force is considered

$$m\ddot{x} + \beta\dot{x} + f(x) = \sigma\,\xi(t) \tag{B1}$$

The stiffness force relation for the symmetric bilinear spring is

$$f(x) = \begin{cases} k_2(x + x_0) & (x < -x_b) \\ k_1 x & (-x_b \leqq x \leqq x_b) \\ k_2(x - x_0) & (x_b < x) \end{cases} \tag{B2}$$

where

$$\gamma^2 = \frac{k_1}{k_2}$$

$$x_0 = (1 - \gamma^2)x_b$$

The bilinear stiffness force relation is plotted as a function of the displacement in figure 1. The excitation random process is stationary Gaussian white noise with zero mean value. The auto-correlation function is a Dirac delta function

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$$\left.\begin{aligned} E\left[\xi(t)\right] &= 0 \\ E\left[\xi(t_1)\ \xi(t_2)\right] &= 2D\ \delta(t_2 - t_1) \end{aligned}\right\} \tag{B3}$$

The power spectral density function of the white noise process has a constant value equal to the quantity D divided by π. The white noise process is multiplied by the amplitude parameter σ, which is a constant in the present development.

The joint probability density function of the displacement and the velocity (conditional on the value of the amplitude parameter σ) is obtained from the Fokker-Planck equation. By using equations (A16) and (A23), the Fokker-Planck equation for the dynamic system of equation (B1) is

$$-v\,\frac{\partial}{\partial x}\left[p(x,v|\sigma)\right] + \frac{\partial}{\partial v}\left\{\left[\beta v + f(x)\right]p(x,v|\sigma)\right\} + D\sigma^2\,\frac{\partial^2}{\partial v^2}\left[p(x,v|\sigma)\right] = 0 \tag{B4}$$

where

$$v = \dot{x}$$

In this case the Fokker-Planck equation can be solved by the method of separation of variables

$$p(x,v|\sigma) = p(x|\sigma)\ p(v|\sigma) \tag{B5}$$

This relation states that the displacement and velocity of the system response are independent. By applying the separation of variables, the Fokker-Planck equation gives separate equations for the two first-order probability density functions. The solution of the equation for the probability density function of the displacement is

$$p(x|\sigma) = C_0 \exp\left[-\frac{\beta}{D\sigma^2}\int_0^x f(\bar{x})\, d\bar{x}\right] \tag{B6}$$

For the bilinear spring (eq. (B2)), the probability density function of the displacement is

$$p(x|\sigma) = \begin{cases} C_0 \exp\left\{-\dfrac{1}{2A_{x2}{}^2\sigma^2}\left[(x + x_0)^2 + \gamma^2 x_b x_0\right]\right\} & (x < -x_b) \\ C_0 \exp\left\{-\dfrac{1}{2A_{x1}{}^2\sigma^2}\, x^2\right\} & (-x_b \leqq x \leqq x_b) \\ C_0 \exp\left\{-\dfrac{1}{2A_{x2}{}^2\sigma^2}\left[(x - x_0)^2 + \gamma^2 x_b x_0\right]\right\} & (x_b < x) \end{cases} \tag{B7}$$

where

$$A_{x1}{}^2 = D/\beta k_1$$

$$A_{x2}{}^2 = D/\beta k_2 = \gamma^2 A_{x1}{}^2$$

The probability density function has Gaussian functional forms in the linear regions of the stiffness force relation, with the variances and mean values for the linear systems corresponding to those regions. The probability density is a continuous function of the displacement as indicated by equation (B6). The unknown constant in equation (B7) is determined by the normalization condition, that is, the total probability must be equal to one

$$C_0{}^{-1} = \sqrt{2\pi} A_{x1}\sigma \left[\operatorname{erf} \eta + g(\gamma, \eta)\right] \tag{B8}$$

where

$$\eta = \frac{x_b}{\sqrt{2}A_{x1}\sigma}$$

$$g(\gamma,\eta) = \gamma \exp\left[(\gamma^2 - 1)\eta^2\right]\operatorname{erfc}(\gamma\eta)$$

The probability density function (eq. (B7)) determines all the moments of the displacement. The mean value and all other odd order moments are zero, a property which follows from the anti-symmetry of the stiffness force. The relation for the variance of the displacement is

$$E\left[x^2|\sigma\right] = \sigma_{xc}^2 = A_x^2(\sigma)\sigma^2 \tag{B9}$$

$$\frac{A_x^2(\sigma)}{A_{x1}^2} = 1 + \frac{(\gamma^2 - 1)\left\{-\frac{2}{\sqrt{\pi}}\eta(\gamma^2 - 1)e^{-\eta^2} + \left[1 + 2(\gamma^2 - 1)\eta^2\right]g(\gamma,\eta)\right\}}{\operatorname{erf}\eta + g(\gamma,\eta)} \tag{B10}$$

The Fokker-Planck equation also gives a differential equation for the probability density function of the system velocity, whose solution is

$$p(v|\sigma) = \frac{1}{\sqrt{2\pi}A_v\sigma}\exp\left(-\frac{v^2}{2A_v^2\sigma^2}\right) \tag{B11}$$

where

$$A_v^2 = \frac{D}{\beta m}$$

Thus the velocity is a Gaussian random variable. The probability density of the velocity is identical to that of a corresponding linear system under white noise excitation, in which case the probability density function is independent of the stiffness force.

APPENDIX C

RESPONSE OF NONSYMMETRIC BILINEAR SPRING SYSTEM

The effects of nonsymmetry in the bilinear spring system are discussed briefly in this appendix. The probability density function of the response of the nonsymmetric dynamic system is obtained from the solution of the Fokker-Planck equation. The solution for the system response is developed first for excitation by the conditional process and then for excitation by the amplitude modulated process. The main item of interest is the mean value of the displacement which can be nonzero due to the nonsymmetry. The differential equation of the system has the same general form as that for the symmetric system (eq. (B1)). The bilinear stiffness force is formed from linear regions, the two outer regions having the same linear stiffness force coefficient. The breakpoints between the linear regions are not symmetric in the displacement:

$$f(x) = \begin{cases} k_2(x + \alpha x_0) & (x < -\alpha x_b) \\ k_1 x & (-\alpha x_b \leqq x \leqq x_b) \\ k_2(x - x_0) & (x_b < x) \end{cases} \qquad \text{(C1)}$$

where

$$\alpha > 0$$

The probability density function of the response to the conditional process is determined by the Fokker-Planck equation. By using the solution for a dynamic system with a general nonlinear stiffness relation (eq. (B6)), the probability density function of the displacement is

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$$\bar{p}(x|\sigma) = \begin{cases} \bar{C}_0 \exp\left\{-\dfrac{1}{2A_{x2}{}^2\sigma^2}\left[(x + \alpha x_0)^2 + \alpha^2\gamma^2 x_b x_0\right]\right\} & (x < -\alpha x_b) \\ \bar{C}_0 \exp\left\{-\dfrac{1}{2A_{x1}{}^2\sigma^2}\, x^2\right\} & (-\alpha x_b \leqq x \leqq x_b) \\ \bar{C}_0 \exp\left\{-\dfrac{1}{2A_{x2}{}^2\sigma^2}\left[(x - x_0)^2 + \gamma^2 x_b x_0\right]\right\} & (x_b < x) \end{cases} \tag{C2}$$

The coefficient is determined by the normalization condition

$$\bar{C}_0{}^{-1} = \sqrt{\frac{\pi}{2}}\, A_{x1}\sigma\left[\operatorname{erf}(\eta) + \operatorname{erf}(\alpha\eta) + g(\gamma,\eta) + g(\gamma,\alpha\eta)\right] \tag{C3}$$

The probability density function of the velocity is unchanged from the symmetric case (eq. (B11)). The joint probability density function of the displacement and velocity, which are independent for the response to the conditional process, is thus known. This function can be used to determine the various moments, the exceedance expression, and other response quantities.

The primary new feature in the nonsymmetric case is the existence of nonzero odd order moments of the displacement. The conditional mean value is obtained from the conditional probability density function (eq. (C2))

$$E[x|\sigma] = \bar{C}_0 A_{x1}{}^2\sigma^2(\gamma^2 - 1)\left\{e^{-\eta^2} - e^{-\alpha^2\eta^2} - \sqrt{\pi}\left[\eta g(\gamma,\eta) - \alpha\eta g(\gamma,\alpha\eta)\right]\right\} \tag{C4}$$

The conditional mean value of the displacement is plotted in figure 23 in nondimensional form as a function of the amplitude parameter σ for one set of the parameters of the nonsymmetric bilinear stiffness force. The conditional mean value shows the properties of the two limiting linear systems. The conditional

mean value approaches zero in the limits either of small values of amplitude parameter σ or of large values of the breakpoints of the bilinear stiffness curve. In these cases the stiffness force is dominated by the inner linear region. In the opposite limits the stiffness force is dominated by the outer regions of the stiffness force relation. For large values of the amplitude parameter the conditional mean value approaches a limiting value, which is obtained from equation (C4),

$$x_{m,\infty} = \lim_{\sigma \to \infty} E[x|\sigma] = \frac{1}{2}(1 - \gamma^2)(1 - \alpha)x_b \qquad (C5)$$

The response of the nonsymmetric dynamic system to excitation by the amplitude modulated process is developed by introducing the random variation of the amplitude parameter. The relation for the mean value of the response to the amplitude modulated process is

$$E[x] = \int_0^\infty E[x|\sigma]\, p(\sigma)\, d\sigma \qquad (C6)$$

The required integration over the amplitude parameter σ is intractable and is done numerically. The mean value of the displacement is plotted in figure 24 in nondimensional form as a function of the standard deviation of the amplitude modulated process. The parameters of the nonsymmetric bilinear stiffness force relation are the same as those used in figure 23. The mean value shows the same qualitative behavior as in the case of the conditional process. In the limit either of small values of the standard deviation of the amplitude process or of large values of the stiffness breakpoints, the mean value approaches zero. In the opposite limits the mean value approaches the same limiting value as in the case of the conditional process (eq. (C5)).

APPENDIX D

LINEAR EQUATIONS OF MOTION FOR THE AIRCRAFT SYSTEM

The linear equations of motion for the aircraft considered are outlined in this appendix. The equations of motion and the aircraft system are described in detail in reference 21. The aircraft is representative of small corporate jet transports. Only the whole-body longitudinal motions of the aircraft are considered. The augmentation system is an autopilot which controls the deflections of the elevator on the horizontal tail.

The equations of motion are basically the classical equations of the dynamic response of a rigid aircraft (ref. 22). The equations are written in the stability axes system with the origin at the airplane mass center. Four coordinates are used: perturbations in nondimensional forward speed, angle of attack, pitch angle, and elevator deflection angle. The equations of motion are written in the form of their Laplace transformation as

$$[M]\{q\} = -\alpha_g\{f\} \tag{D1}$$

where

$$M = \begin{bmatrix} 2\mu \frac{\bar{c}}{2u_0} s - C_{x_u} & -C_{x_\alpha} & C_{L,0} & 0 \\ 2C_{L,0} - C_{z_u} & 2\mu \frac{\bar{c}}{2u_0} s - C_{z_\alpha} - C_{z_{\dot{\alpha}}} \frac{\bar{c}}{2u_0} s\ell(s) & -2\mu \frac{\bar{c}}{2u_0} s - C_{z_q} \frac{\bar{c}}{2u_0} s & -C_{z_\delta} \\ -C_{m_u} & -C_{m_\alpha} - C_{m_{\dot{\alpha}}} \frac{\bar{c}}{2u_0} s\ell(s) & i_B \frac{\bar{c}^2}{4u_0^2} s^2 - C_{m_q} \frac{\bar{c}}{2u_0} s & -C_{m_\delta} \\ 0 & 0 & \frac{1}{st_{ch} + 1}(K_\theta + K_{\dot{\theta}} s) & -1 \end{bmatrix}$$

$$q = \begin{Bmatrix} \hat{u} \\ \alpha \\ \theta \\ \delta \end{Bmatrix}$$

and

$$f = \begin{Bmatrix} C_{x_\alpha} \\ C_{z_\alpha} + \left(C_{z_{\dot{\alpha}}} - C_{z_q}\right)\frac{\bar{c}}{2u_0}\, s\ell(s) \\ C_{m_\alpha} + \left(C_{m_{\dot{\alpha}}} - C_{m_q}\right)\frac{\bar{c}}{2u_0}\, s\ell(s) \\ 0 \end{Bmatrix}$$

The classical equations are modified to include a pure convective time lag of the wing downwash and of the turbulence velocity between the wing and the horizontal tail. This modification is achieved by multiplying the angle-of-attack rate and the turbulence rate terms by the function

$$\ell(s) = \frac{1}{\tau s}(1 - e^{-\tau s}) \qquad \text{(D2)}$$

The quantity τ is the distance between the aircraft center of mass and the aerodynamic center of the tail divided by the forward speed of the aircraft. For this application it is assumed that the terms associated with the angle-of-attack rate $\dot{\alpha}$ and the pitch rate q in the equations of motion originate from only the tail forces. Except for the transport time lag between the wing and tail, the unsteady aerodynamic effects are represented in

quasi-steady form. Except for the notation given in equation (D2), the notation of equation (D1) follows that of reference 21.

The autopilot control relation consists of first-order servosystem dynamics and two feedback gains, one for the pitch K_θ and the other for the pitch rate $K_{\dot{\theta}}$. Therefore,

$$\delta = \frac{1}{st_{ch} + 1}\left(K_\theta + K_{\dot{\theta}} s\right)\theta \qquad \text{(D3)}$$

The present study considers only one set of the parameters of the control law. The servosystem characteristic time t_{ch} has a value of 0.037 second. The pitch gain K_θ has a value of 1.0; the pitch-rate gain $K_{\dot{\theta}}$ has a value of 0.054. This value of pitch-rate gain corresponds to the value of 10.0 quoted in reference 21, where a time scale factor (mean aerodynamic chord of the wing divided by twice the aircraft forward speed) was omitted.

The airplane equations of motion are solved for the variances of the response by frequency response methods. The stationary random process of the vertical component of the one-dimensional turbulence velocity field is represented by the von Karman form of the power spectral density function (ref. 7) rather than the Dryden form used in reference 21. A value of 762 m (2500 ft) is used for the scale of turbulence. The standard deviation factor of the excitation process has unit variance. The power spectral density functions of the response are determined from the product of the spectral function of the vertical component of the turbulence velocity and the square of the modulus of the frequency response functions which are obtained from equation (D1). The variances of the response are obtained by integrating the power spectral density functions over an appropriate range of frequency values. The flight condition considered has an altitude of 6100 m (20 000 ft) and a Mach number of 0.75. The mass data and the stability derivatives for this condition (condition IV) are listed in reference 21.

The equations of motion of the aircraft can be presented in a simpler, approximate form which is used to study the effects of

the turbulence and the control system upon the pitching motion. The approximate relation is the same as equation (D1) except that the forward-speed coordinate and the convective time lag between the wing and tail are omitted. The resulting equation for the pitch coordinate is

$$\left(b_2 s^2 + b_1 s + b_0\right)\theta = -\left(a_1 + \frac{a_0}{s}\right)\alpha_g - \left(d_1 + \frac{d_0}{s}\right)\delta \tag{D4}$$

where

$$b_2 = i_B\left(2\mu - C_{z_{\dot{\alpha}}}\right)(t^*)^2$$

$$b_1 = -\left[i_B C_{z_\alpha} + 2\mu\left(C_{m_{\dot{\alpha}}} + C_{m_q}\right) + C_{m_{\dot{\alpha}}} C_{z_q} - C_{m_q} C_{z_{\dot{\alpha}}}\right]t^*$$

$$b_0 = C_{z_\alpha} C_{m_q} - C_{m_\alpha}\left(2\mu + C_{z_q}\right)$$

$$d_1 = \left[-C_{m_\delta}\left(2\mu - C_{z_{\dot{\alpha}}}\right) - C_{z_\delta} C_{m_{\dot{\alpha}}}\right]t^*$$

$$d_0 = \left(C_{m_\delta} C_{z_\alpha} - C_{z_\delta} C_{m_\alpha}\right)$$

$$t^* = \frac{\bar{c}}{2u_0}$$

In this form the excitation function consists of the aerodynamic forces due to both the turbulence field and the elevator deflection. The coefficients associated with the elevator deflection (that is, d_0 and d_1) have positive values for the flight condition considered in the present study. The constants a_0 and a_1 for the turbulence field are similar to the constants d_0 and d_1 for the elevator deflection.

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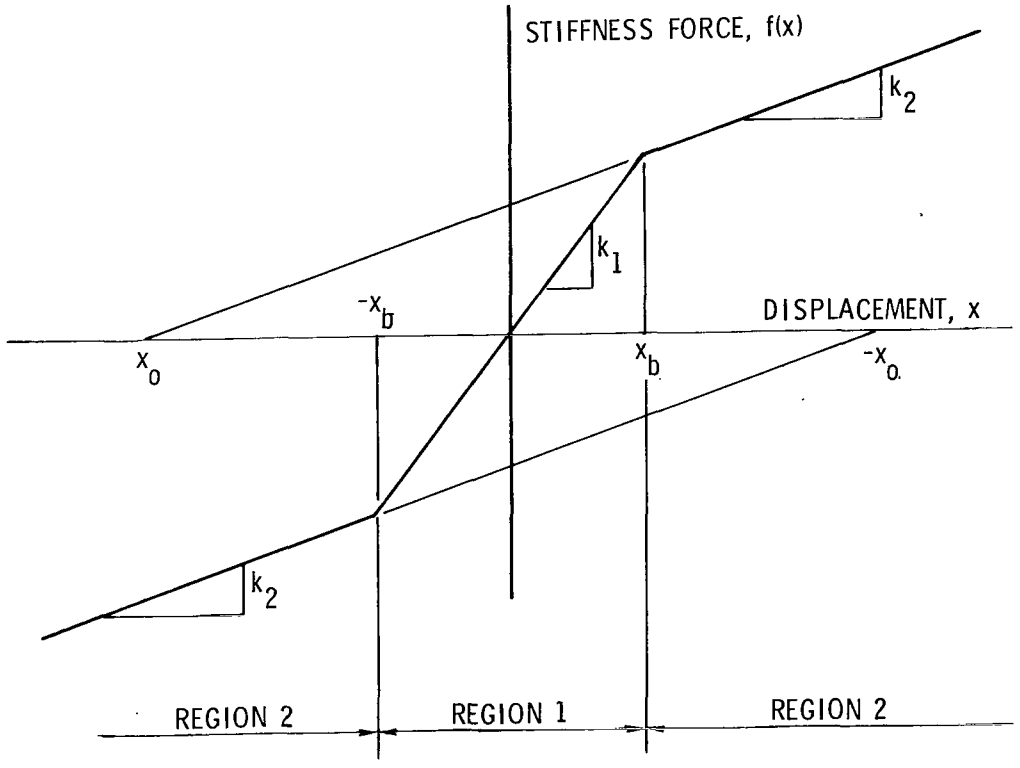

Figure 1.- Symmetric bilinear stiffness force relation.

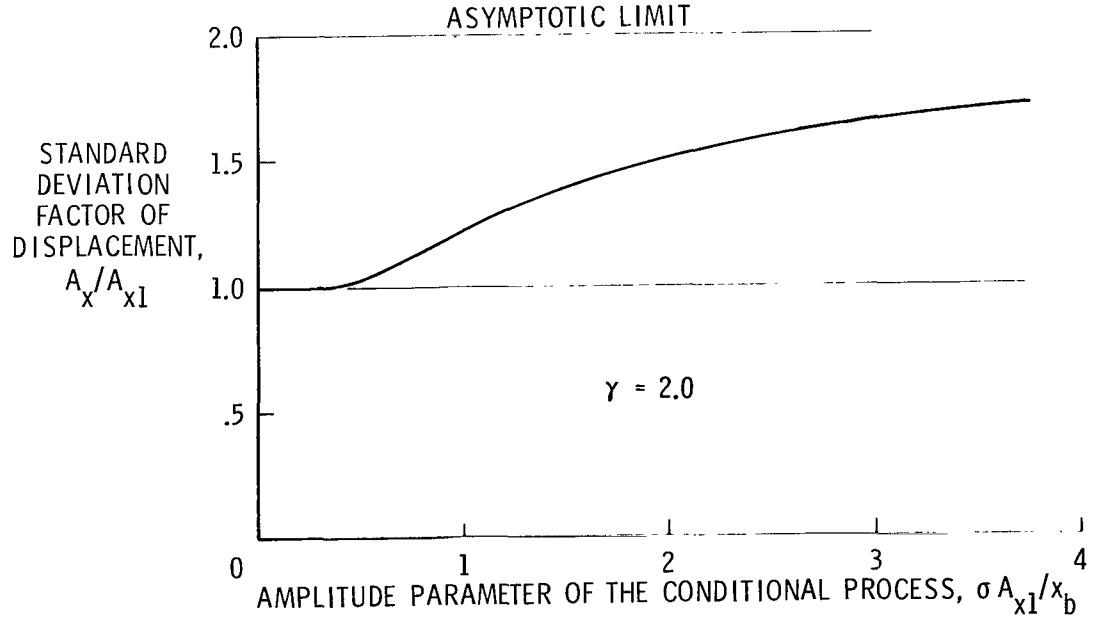

Figure 2.- Standard deviation factor of displacement response to conditional process.

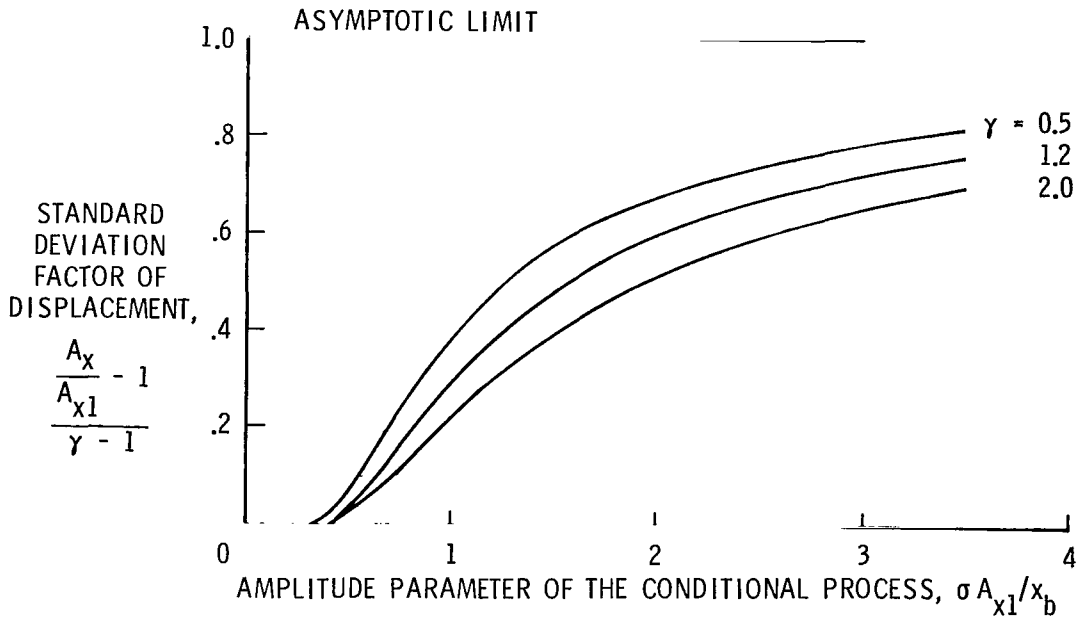

Figure 3.- Standard deviation factor of displacement response to conditional process, normalized form.

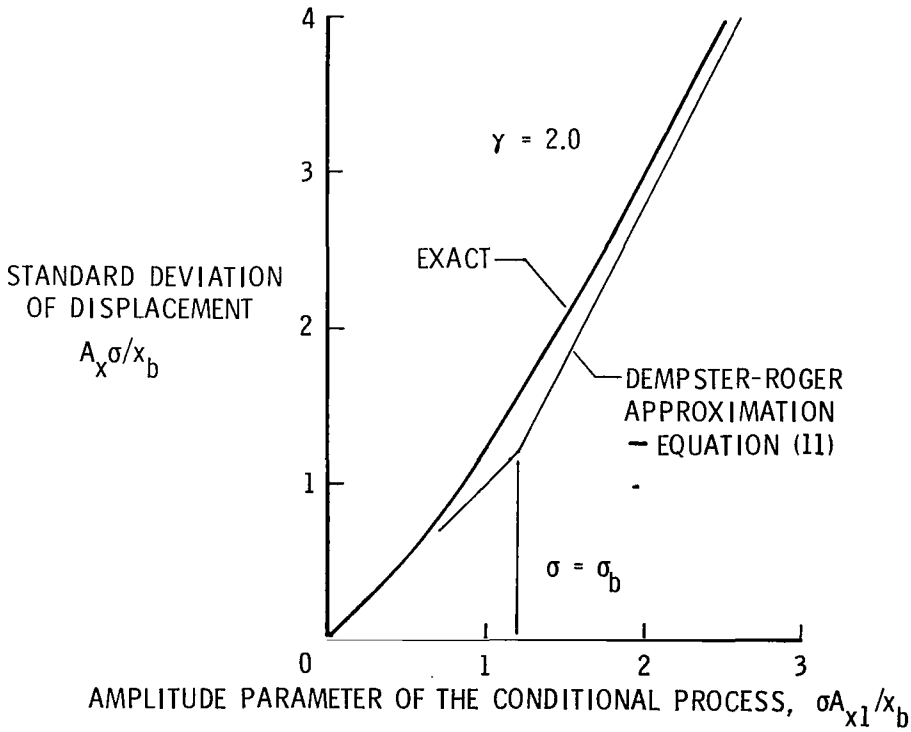

Figure 4.- Standard deviation of displacement response to conditional process.

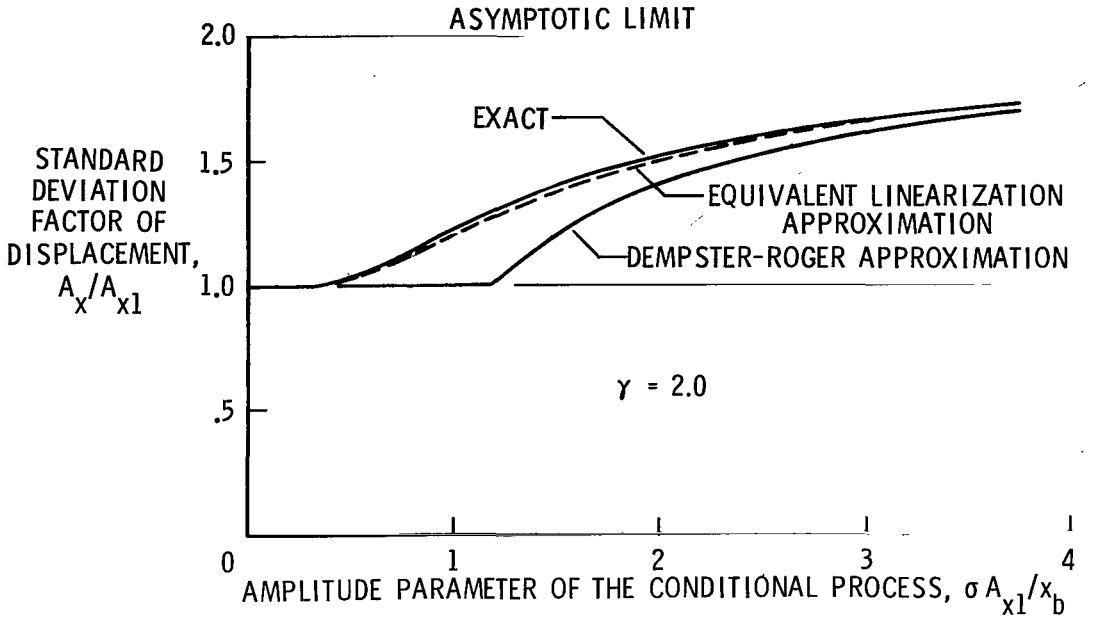

Figure 5.- Standard deviation factor of displacement response to conditional process, exact and approximate results.

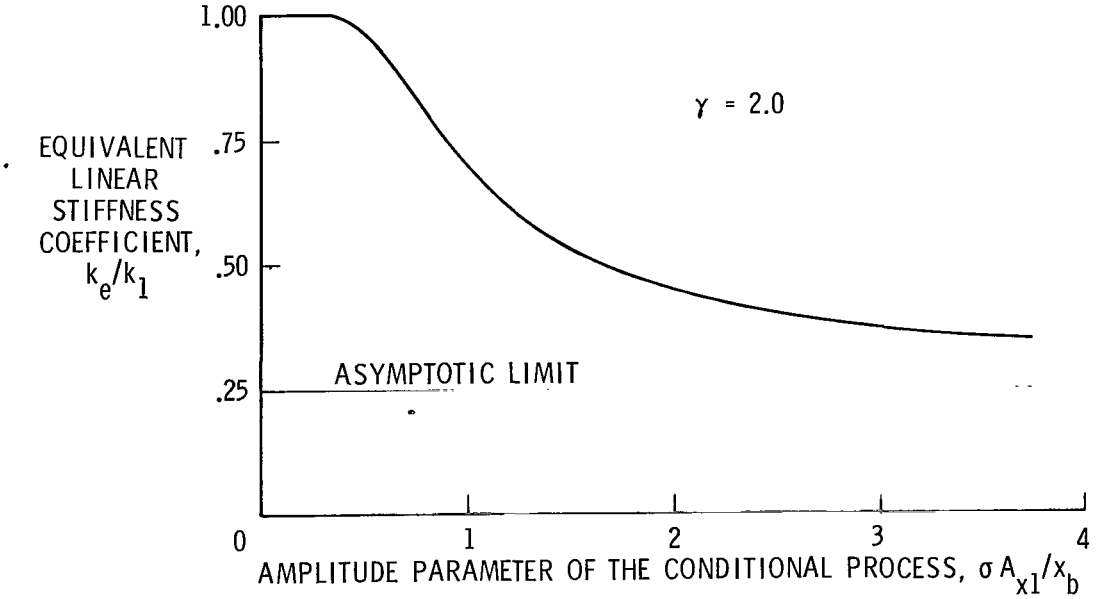

Figure 6.- Equivalent linear stiffness coefficient for response to conditional process.

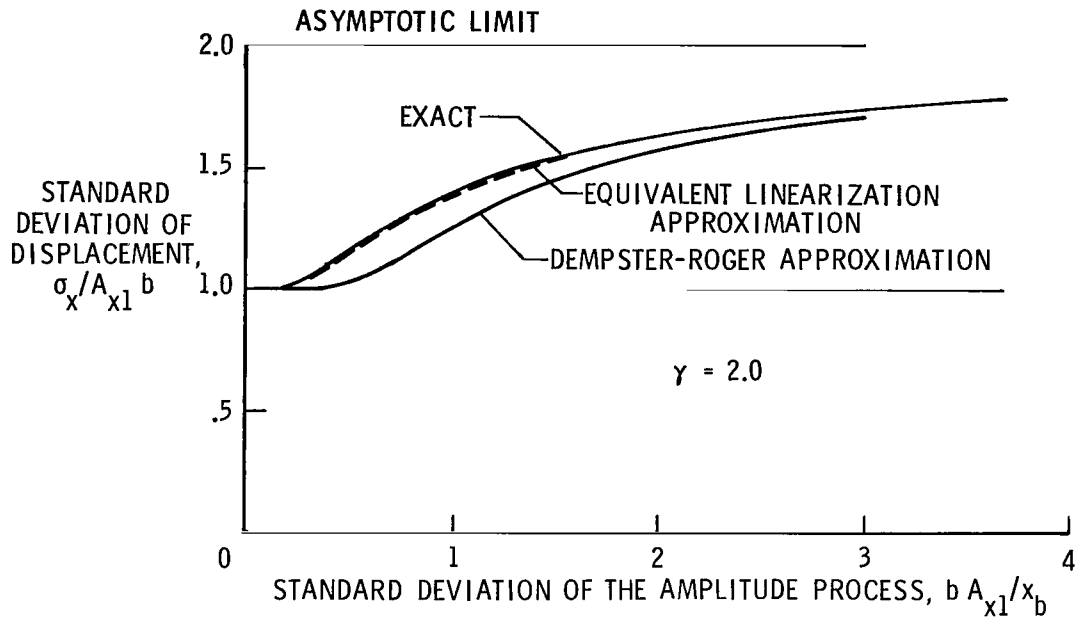

Figure 7.- Standard deviation of displacement response to amplitude modulated process, exact and approximate results.

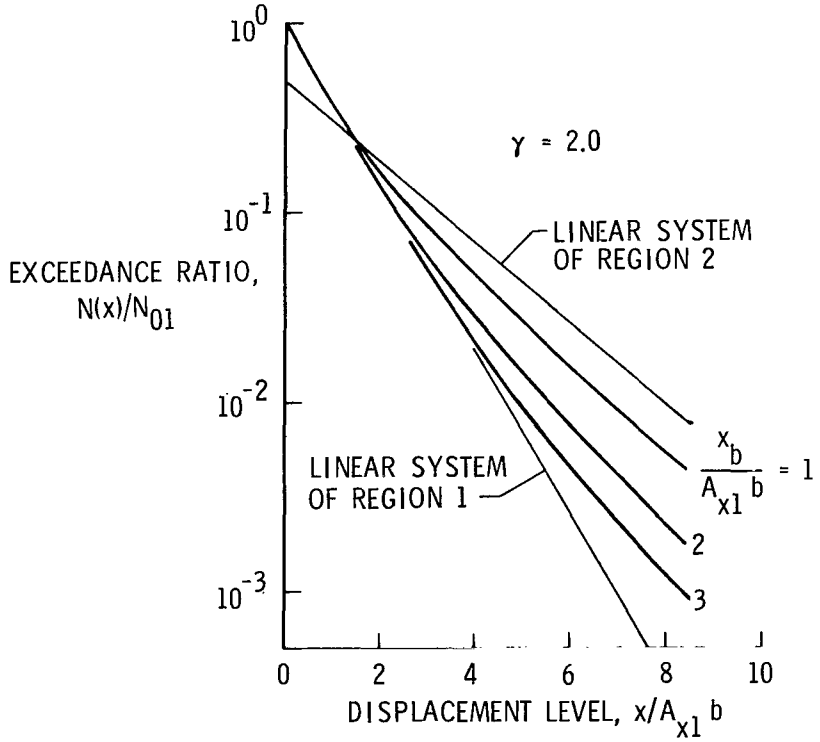

Figure 8.- Exceedance ratio for displacement response to amplitude modulated process.

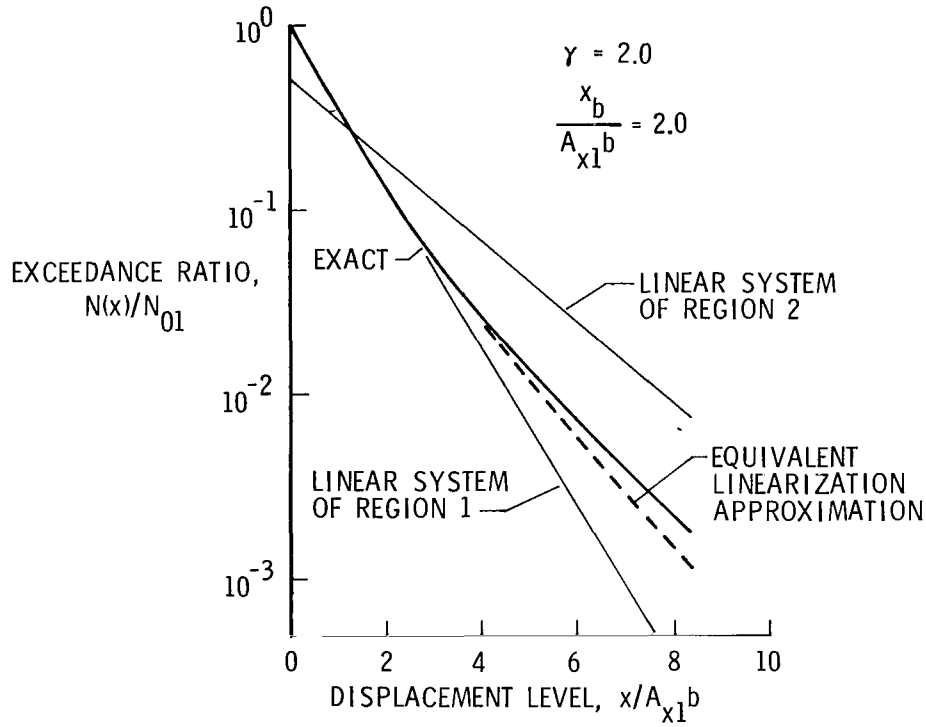

Figure 9.- Exceedance ratio for displacement response to amplitude modulated process, exact and approximate results.

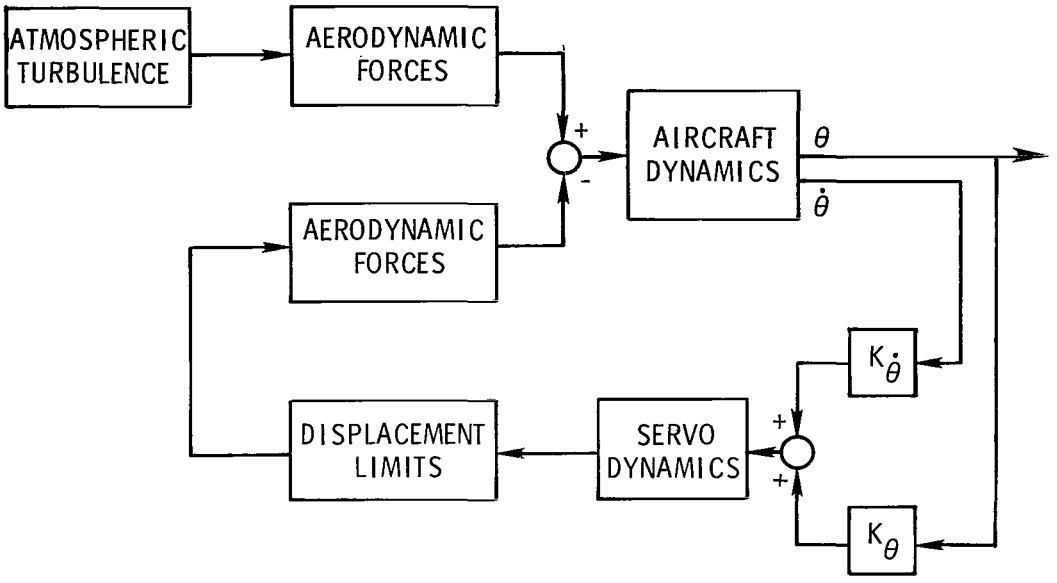

Figure 10.- Block diagram of basic elements of augmented aircraft system.

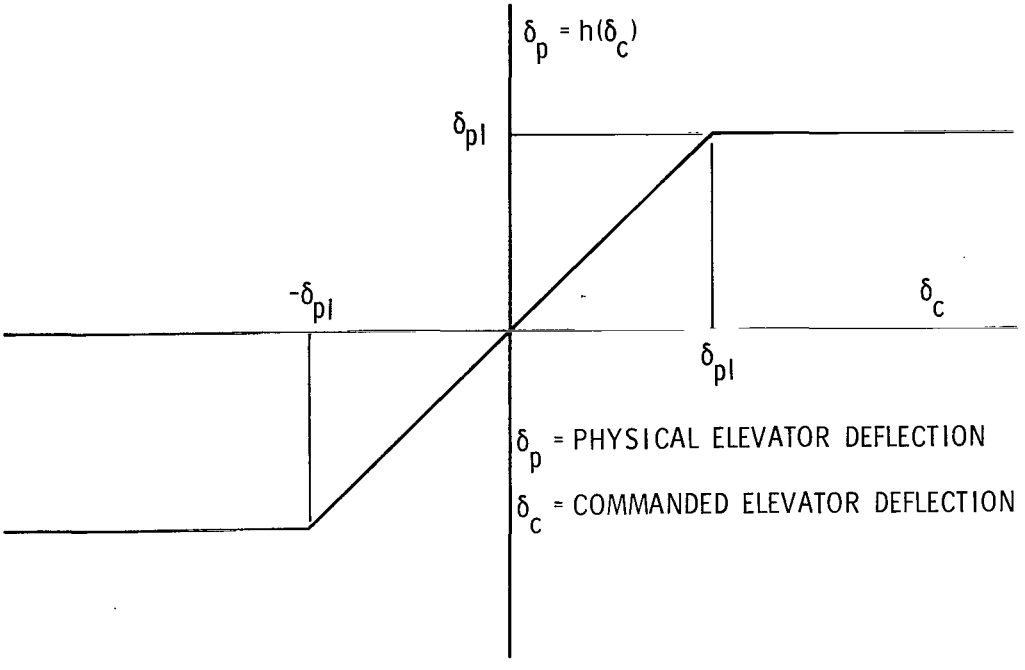

Figure 11.- Relation between physical and commanded elevator deflections.

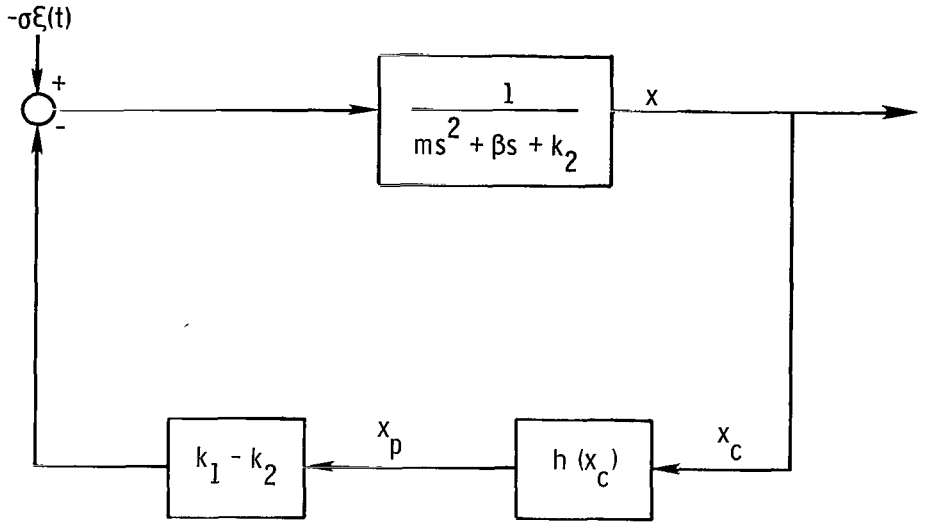

Figure 12.- Block diagram of bilinear spring system.

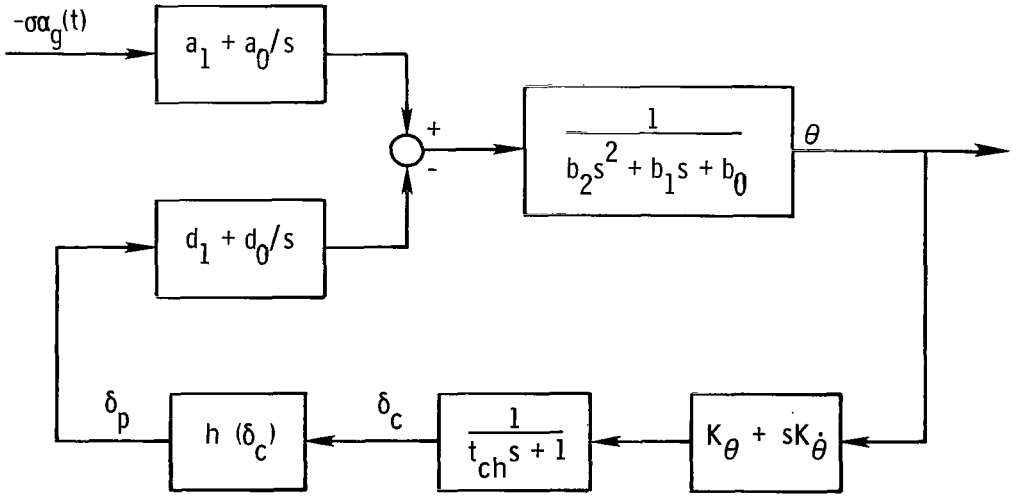

Figure 13.- Block diagram of augmented aircraft system.

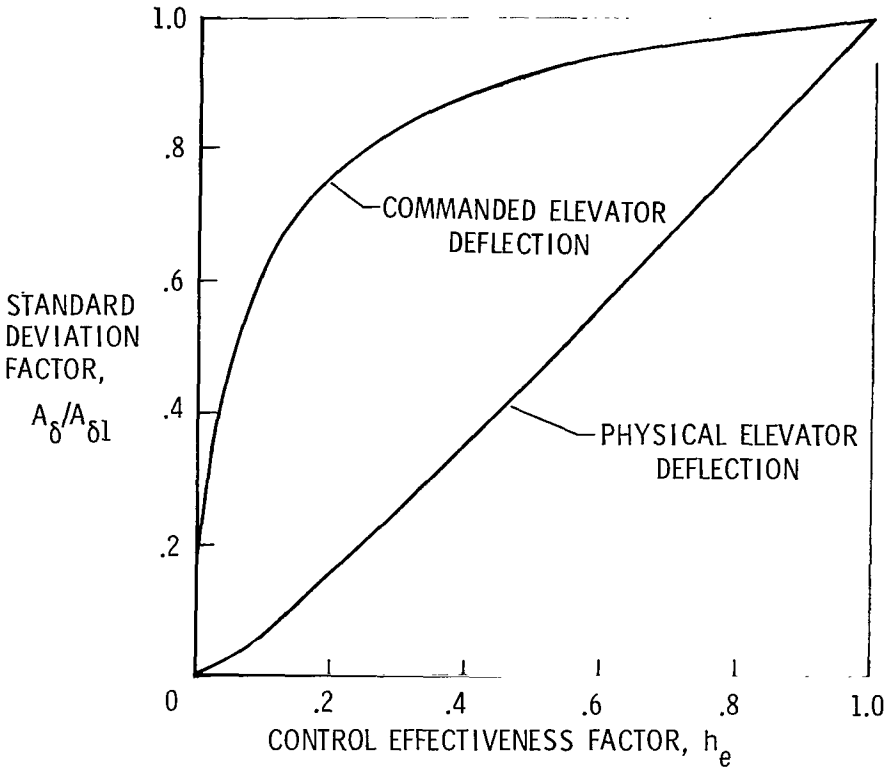

Figure 14.- Relation between standard deviation factors of elevator deflection and control effectiveness factor.

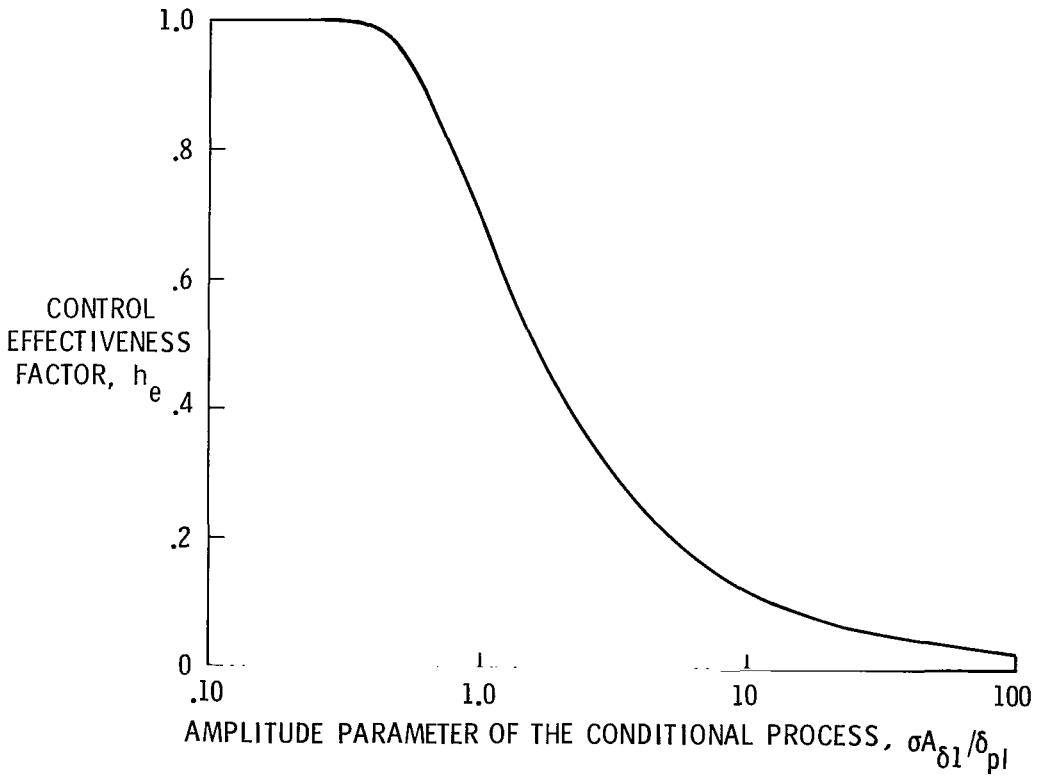

Figure 15.- Relation between control effectiveness factor and amplitude parameter of conditional process.

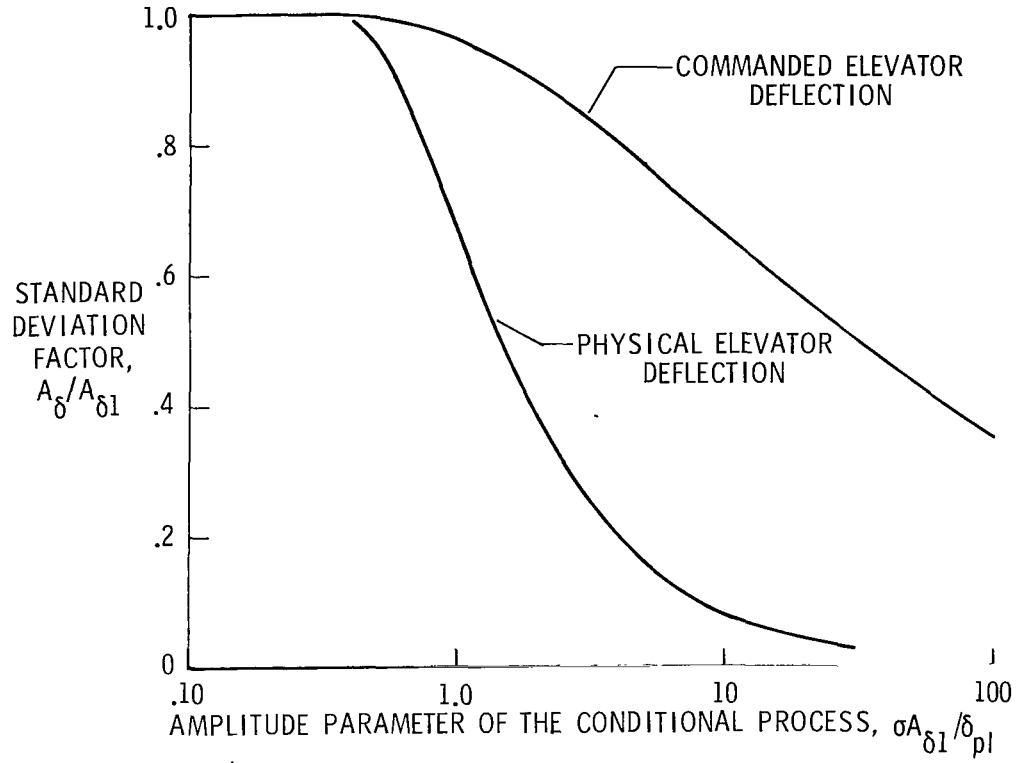

Figure 16.- Standard deviation factors of elevator deflection for conditional process.

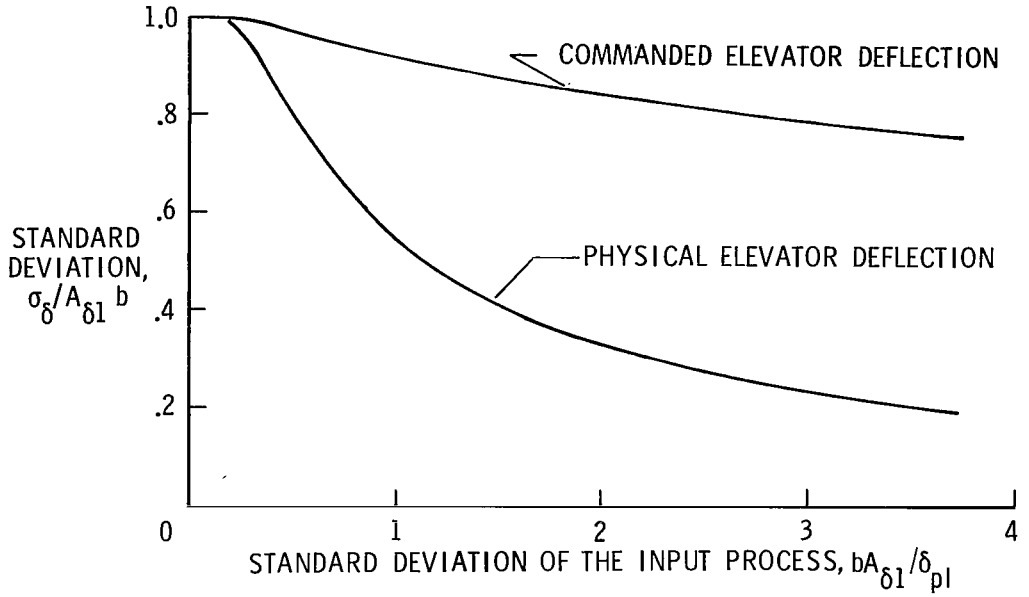

Figure 17.- Standard deviations of elevator deflections for amplitude modulated process.

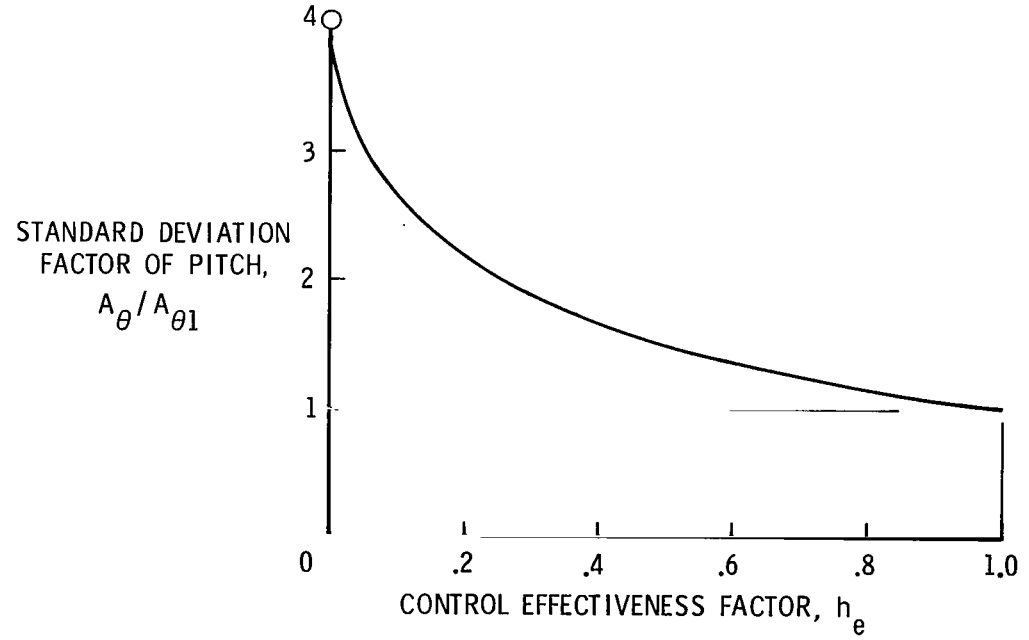

Figure 18.- Relation between standard deviation factor of pitch and control effectiveness factor.

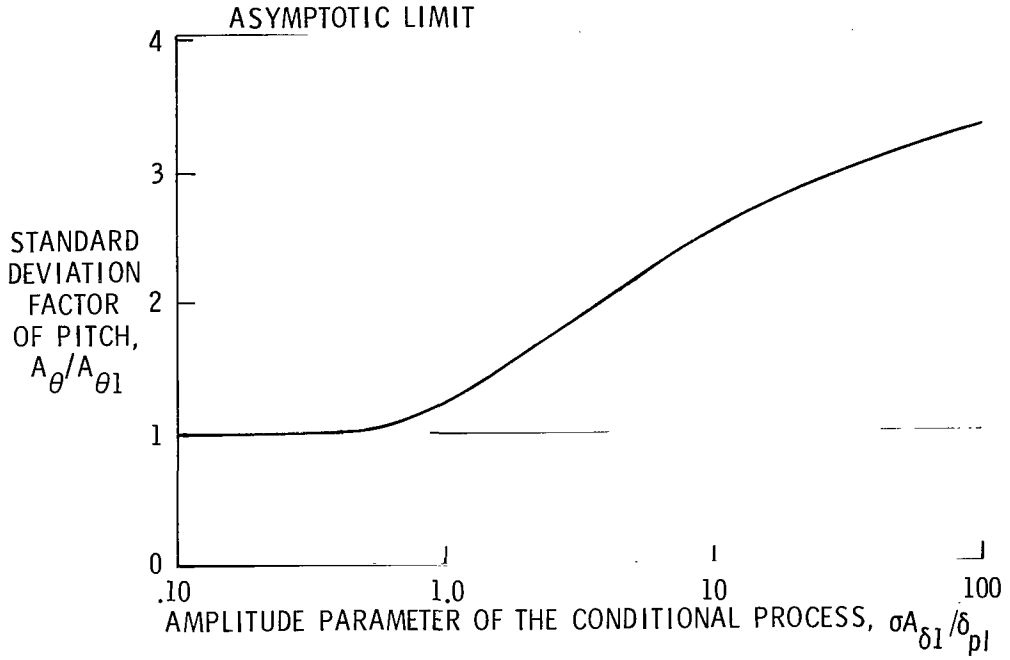

Figure 19.- Standard deviation factor of pitch for conditional process.

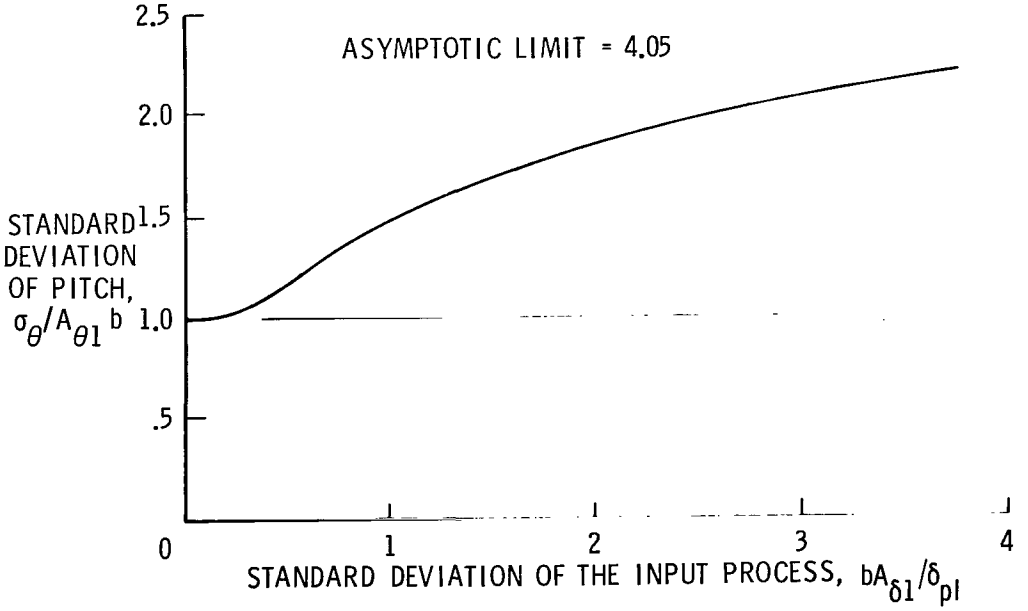

Figure 20.- Standard deviation of pitch for amplitude modulated process.

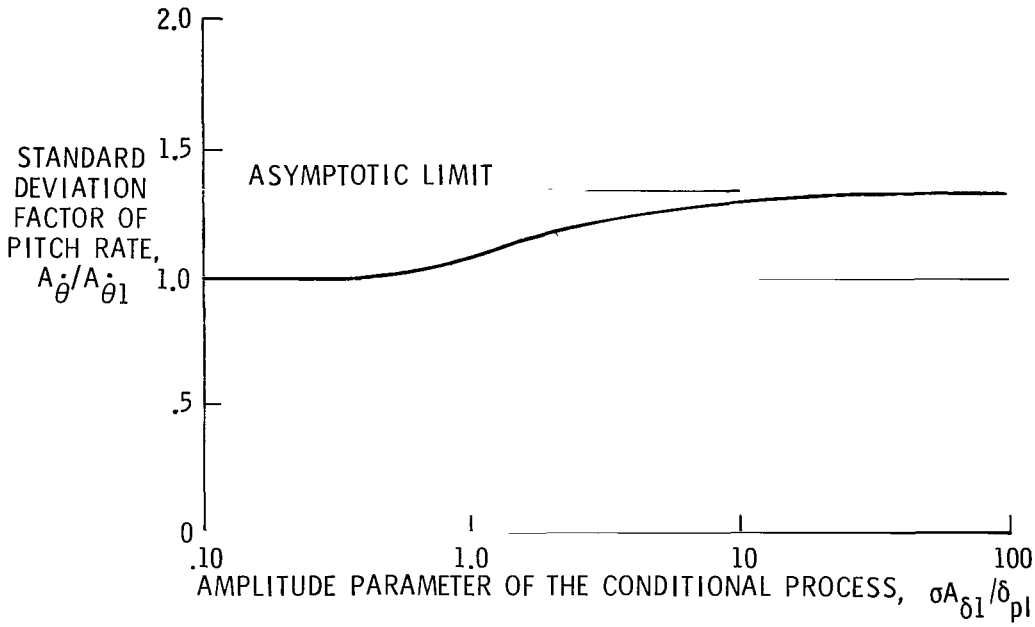

Figure 21.- Standard deviation factor of pitch rate for conditional process.

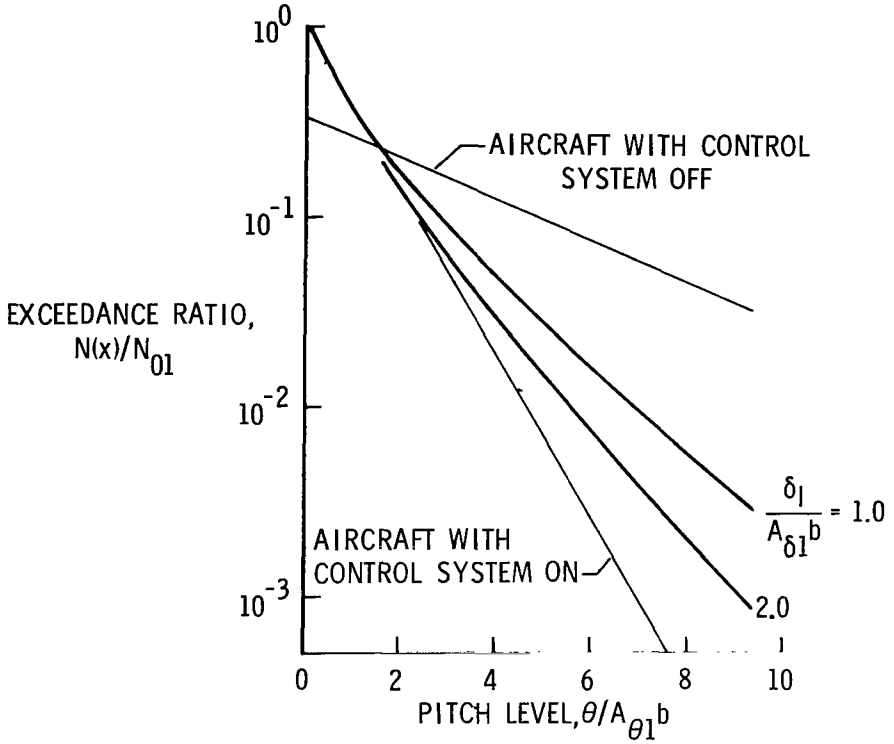

Figure 22.- Exceedance ratio for pitch response to amplitude modulated process.

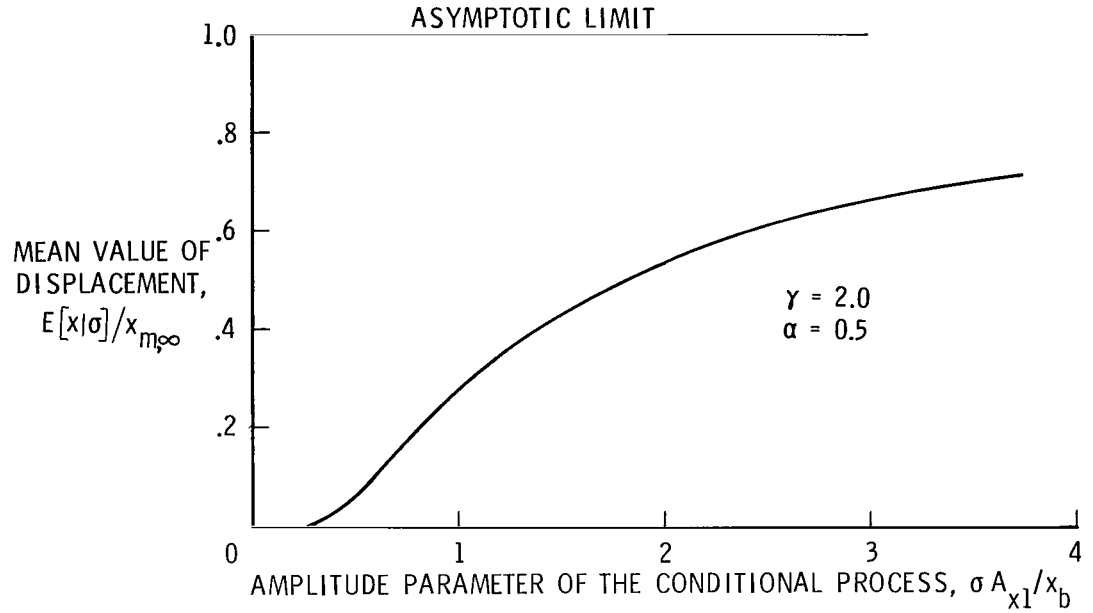

Figure 23.- Mean value of displacement response to conditional process, nonsymmetric system.

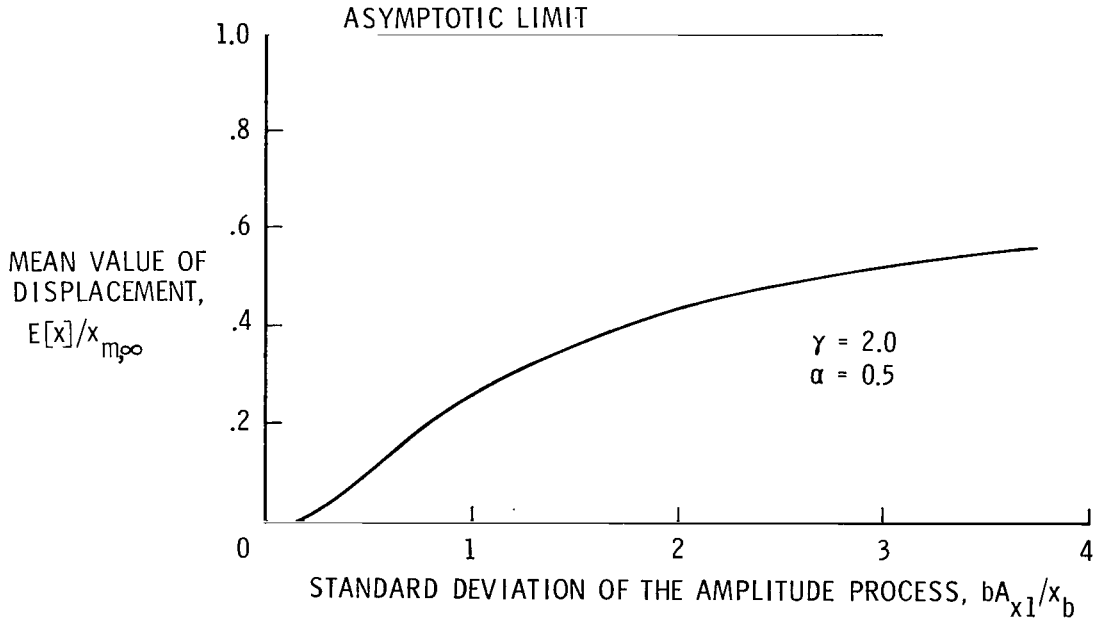

Figure 24.- Mean value of displacement response to amplitude modulated process, nonsymmetric system.

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